MAT 215B HW02

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Due Friday at 9 pm on Gradescope

Instructions Your solution **must be typeset**, but you may include handdrawn figures. Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours. Some of the exercises in HWs come from Hatcher or are slightly modified versions of Hatcher exercises. For example, 'HE 2.1.1' means 'Hatcher Chapter 2 Section 1 Exercise 1'. Each exercise will be graded out of 3 points.

Exercise 1

Explain why $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ for any space $X \neq \emptyset$.

You may use the following lemma:

Lemma. If $A \xrightarrow{\alpha} \mathbb{Z}$ is a surjective homomorphism of $\mathbb{Z}$-modules, then $A \cong \ker \alpha \oplus \mathbb{Z}$.

This fact is implicitly used in the text. This is however not necessarily true if we replace $\mathbb{Z}$ with other $\mathbb{Z}$-modules! For example, we have a surjective map $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/2\mathbb{Z}$ with $\ker(f) \cong \mathbb{Z}/2\mathbb{Z}$. However, $\mathbb{Z}/4\mathbb{Z}$ is *not* isomorphic to the Klein-4 group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (which has no order-4 element).

Exercise 2

It is easy to understand maps (e.g. singular simplicies) into a single-point space $X = \{*\}$. Prove that $\widetilde{H}_n(X) = 0$ for all n by writing down the singular chain complex and computing the homology. This is in the text as well. Use your own words.

Exercise 3

Prove that chain homotopic maps induce the same map on homology.

Exercise 4

(Hatcher §2.1 Ex 11) Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Exercise 5

Complete the proof that the LES associated to the SES

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

is indeed exact, by proving the two remaining statements not in the notes:

(a) $\ker \partial \subseteq \operatorname{im} j_*$

(b) $\ker i_* \subseteq \operatorname{im} \partial$

Exercise 6

For an exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

show that $C = 0$ iff α is surjective and δ is injective.

This is the first half of §2.1 Ex 15. The second half of the exercise states, "Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n." We will talk about this in class.