MAT 215B HW03

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Due Friday at 9 pm on Gradescope

Instructions Your solution **must be typeset**, but you may include handdrawn figures. Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours. Some of the exercises in HWs come from Hatcher or are slightly modified versions of Hatcher exercises. For example, 'HE 2.1.1' means 'Hatcher Chapter 2 Section 1 Exercise 1'. Each exercise will be graded out of 3 points.

Exercise 1

(a) Determine the module A and the maps f and g that make the following sequence exact:

$$0 \to \mathbb{Z} \xrightarrow{(\text{id},0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0$$

(b) Find two non-isomorphic modules A_1, A_2 that can take the place of A in the sequence below, along with maps that make the sequence exact:

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Exercise 2

For a space X, let SX denote the *free suspension* of X, which is thought of as two cones CX of X glued together along their boundaries, which are homeomorphic to X. Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n by constructing an explicit chain map $s = \{s_n : C_n(X) \to C_{n+1}(SX)\}$ that induces isomorphisms $\widetilde{H}_n(X) \xrightarrow{\cong} \widetilde{H}_{n+1}(SX)$.

Exercise 3

Use the LES for the homology of a pair to prove that

$$H_\bullet(X, \{x_0\}) \cong \widetilde{H}_\bullet(X)$$

where x_0 is a point in $X \neq \emptyset$.

Exercise 4

(a) Compute $H_\bullet(X, A)$ where $X = S^2$ and A is a set of k points in X.

(b) Compute $H_\bullet(X, A)$ where $X = S^1 \times S^1$ and A is one of the S^1's. In other words, X is a torus and A is a meridian.

Exercise 5

This exercise isn't exactly quick, but the solution is in the book (Hatcher, bottom of page 129), and the point is that you should work through it. This will be graded for completion, and you do not need to type up your notes on this proof if writing by hand is the way you like to check proofs.

We will need the following lemma in our proof that $H_\bullet^\Delta(X) \cong H_\bullet(X)$.

Lemma. (The Five-Lemma) Consider a commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

where the two rows are exact. If α, β, δ, and ε are isomorphisms, then γ is also an isomorphism.

In this exercise you will prove this lemma, along with some more general statements.

(a) Suppose β and δ are surjective, and ε is injective. Prove that γ is surjective.

(b) Suppose β and δ are injective, and α is surjective. Prove that γ is injective.

(c) Prove the Five-Lemma.

Exercise 6

This exercise introduces the mapping cone construction in homological algebra.

Let $(\mathcal{C}, \partial)$ and $(\mathcal{C}', \partial')$ be two chain complexes, and let $f : \mathcal{C} \to \mathcal{C}'$ be a chain map. We define the *mapping cone of f* as the chain complex $(\mathrm{Cone}(f), D)$ given by the following:

- $\mathrm{Cone}(f)_n = C_n \oplus C'_{n+1}$
- $D_n = \begin{pmatrix} -\partial_n & 0 \\ f_n & \partial'_{n+1} \end{pmatrix}$

(a) Prove that $\mathrm{Cone}(f)$ is indeed a chain complex.

(b) Suppose the chain map f induces an isomorphism on homology, i.e. $(f_*)_n : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ is an isomorphism for all n. Prove that $\mathrm{Cone}(f)$ is *acyclic*, i.e. $H_n(\mathrm{Cone}(f)) = 0$ for all n.