MAT 215B HW04

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Due Friday at 9 pm on Gradescope

Instructions Your solution **must be typeset**, but you may include handdrawn figures. Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours. Some of the exercises in HWs come from Hatcher or are slightly modified versions of Hatcher exercises. For example, 'HE 2.1.1' means 'Hatcher Chapter 2 Section 1 Exercise 1'. Each exercise will be graded out of 3 points.

Exercise 1

We used this fact in class.

Suppose we have two pairs of spaces $A \subseteq X$ and $B \subseteq Y$, and a map $f : X \to Y$ such that $f(A) \subseteq B$. Furthermore, suppose that f induces a homotopy equivalence from X to Y, and $f|_A$ induces a homotopy equivalence from A to B. Prove that $f_* : H_\bullet(X, A) \to H_\bullet(Y, B)$ is an isomorphism.

Hint: Use LESs of pairs!

Exercise 2

(HE 2.1.29) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Exercise 3

(HE 2.1.26) Let $X = [0, 1]$ and $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\} \subset X$. Show that $H_1(X, A) \ncong \widetilde{H}_1(X/A)$.

Use may use the following fact which we will discuss later in the course:

Fact. For any space X, $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$, i.e.

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)],$$

where $[\pi_1(X), \pi_1(X)]$ is the commutator subgroup. You may also use without proof the fact stated in Example 1.25 that π_1 of the shrinking wedge of circles is uncountably generated. Compare with Proposition 2.22 and see Example 1.25.

Exercise 4

A *knot* $K \subset S^3$ is the image of an embedding of $S^1 \hookrightarrow S^3$. Suppose a knot K has a closed neighborhood N homeomorphic to $S^1 \times D^2$, called a *tubular neighborhood*. Then ∂N is an embedded $S^1 \times S^1 \hookrightarrow S^3$. The *knot complement* is $S^3 \setminus \mathring{N}$, a compact manifold with torus equal to ∂N.

Using the exact sequence of the pair (S^3, N) and excision, calculate the relative homology groups of the knot complement rel boundary, $H_n(S^3 \setminus \mathring{N}, \partial N)$, for all n.