MAT 215B HW05

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Due Friday at 9 pm on Gradescope

Instructions Your solution **must be typeset**, but you may include handdrawn figures. Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours. Some of the exercises in HWs come from Hatcher or are slightly modified versions of Hatcher exercises. For example, 'HE 2.1.1' means 'Hatcher Chapter 2 Section 1 Exercise 1'. Each exercise will be graded out of 3 points.

Exercise 1

Compute the homology for the following spaces:

(a) Σ_g, the closed, connected, orientable surface of genus g

(b) N_k, the closed, connected, nonorientable surface with nonorientable genus k.

Exercise 2

Recall that the *real n-dimensional projective space* $\mathbb{RP}^n$ is the quotient of S^n by the antipodal map $-\mathbb{1}$. Alternatively, we may think of $\mathbb{RP}^n$ as the moduli space of lines in $\mathbb{R}^{n+1}$ through the origin. Hence $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\sim$ where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for all $\lambda \in \mathbb{R} - \{0\}$. Equivalence classes are represented by *homogeneous coordinates*: we write $[x_0 : x_1 : \cdots : x_n]$ where not all x_i are 0, where $[x_0 : x_1 : \cdots : x_n] \sim [\lambda x_0 : x_1 : \cdots : \lambda x_n]$ is understood.

Define

$$X = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = 1\} \qquad \text{and} \qquad Y = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = 0\}.$$

Convince yourself that $\mathbb{RP}^n = X \sqcup Y$, and that $X \cong \mathbb{R}^n$ and $Y \cong \mathbb{RP}^{n-1}$.

(a) Show that $\mathbb{RP}^n$ has a cell decomposition with one k-cell e^k for each $0 \leq k \leq n$. Thus the k-skeleton is homeomorphic to $\mathbb{RP}^k$.

(b) Calculate the degree of the attaching map that glues e^k to the $(k-1)$-skeleton.

(c) Compute the cellular homology of $\mathbb{RP}^n$.

Exercise 3

The *complex n-dimensional projective space* $\mathbb{CP}^n$ is defined as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$$

for all $\lambda \in \mathbb{C} - \{0\}$. As in the real projective case, we use homogeneous coordinates $[z_0 : \ldots : z_n]$ where not all $z_i = 0$, with the equivalence $\sim$ understood.

(a) Show that $\mathbb{CP}^n$ has a cell decomposition with one k-cell for each *even* integer from 0 to $2n$.

(b) Compute the cellular homology of $\mathbb{CP}^n$.

Exercise 4

Consider the 2-complex $X = S^1 \times (S^1 \vee S^1)$.

(a) Find a cell decomposition for X and calculate the associated degrees of the gluing maps.

(b) Compute the homology of X.