MAT 215B HW06

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Due **Monday**, March 9th, 2026 at 9 pm on Gradescope

Instructions Your solution **must be typeset**, but you may include handdrawn figures. Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours. Some of the exercises in HWs come from Hatcher or are slightly modified versions of Hatcher exercises. For example, 'HE 2.1.1' means 'Hatcher Chapter 2 Section 1 Exercise 1'. Each exercise will be graded out of 3 points.

Exercise 1

(Hatcher §2.2 Ex 28) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

Exercise 2

Use the Mayer–Vietoris sequence to compute the homology of the space X obtained by gluing a Möbius band along its boundary to the circle $\mathbb{RP}^1$ inside $\mathbb{RP}^2$.

Exercise 3

Compute the reduced singular homology of $S^1 \times S^2$ using

(a) Künneth's formula

(b) the reduced Mayer–Vietoris sequence

Hint: Write $S^1 \times S^2$ as the union of two copies of $S^1 \times D^2$.

Exercise 4

(Hatcher §3.A Ex 1) Use the Universal Coefficient Theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated (so the Euler characteristic $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X;\mathbb{Z})$ is defined), then for any coefficient field F we have

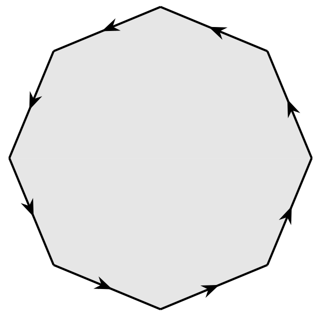
$$\chi(X) = \sum_n (-1)^n \dim H_n(X;F).$$

You may use the following fact:

Fact. $\dim_F \operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, F) = \dim_F \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} F$.

Exercise 5

Let $n \geq 3$. Consider the space X obtained by identifying all 8 sides of an octagon as shown:



(a) Compute the cellular homology $H_*(X; R)$ of X with coefficients in $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_2$, and $\mathbb{F}_3$.

(b) Compute the cellular cohomology $H^*(X; R)$ of X with coefficients in $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_2$, and $\mathbb{F}_3$.