MAT 215B
Final Take-Home Exam-Homework

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Due **Wednesday**, March 18th, 2026 at 9 pm on Gradescope

Instructions This is an open-notes, open-book take-home final exam. All policies agree with the usual policies for homeworks. In particular, you are encouraged to collaborate with others but you must understand and write up your own solutions.

Submit your typeset solutions to Gradescope by the due date and time. You may hand-draw any figures.

Exercise 1

Use the Mayer–Vietoris sequence to show that for a space X, the suspension ΣX satisfies $\widetilde{H}_n(\Sigma X) \cong \widetilde{H}_{n-1}(X)$ for all n.

Exercise 2

In this problem, you will compute the homology groups of $X = S^1 \times (S^1 \vee S^1)$ in different ways.

(a) Describe a cell decomposition for X and compute the cellular homology of X.

(b) Compute $H_*(X)$ using the Künneth formula.

(c) Compute $H_*(X)$ using a Mayer–Vietoris sequence, by viewing X as $A \cup B$ where A and B are homotopic to tori and $A \cap B \simeq S^1$.

Exercise 3

Find a handle decomposition for S^2 with four handles. Draw an embedded $S^2 \subset \mathbb{R}^3$ such that the Morse function $(x, y, z) \mapsto z$ induces this handle decomposition.

Exercise 4

Find a handle decomposition for the 3-torus $T^3 = S^1 \times S^1 \times S^1$.

Hint: We may view T^3 as the identification space $[0,1]^3 / \sim$ where $\sim$ identifies opposite faces:

- $(0, y, z) \sim (1, y, z)$,

- $(x, 0, z) \sim (x, 1, z)$, and

- $(x, y, 0) \sim (x, y, 1)$.

What happens when you consider a radial Morse function starting at the center of the cube?

Exercise 5

Let M be a smooth manifold of dimension n, and let TM denote its tangent bundle. Recall that TM is a rank n bundle (fibers $\cong \mathbb{R}^n$) over M.

A *(smooth) vector field* V on M is a nonvanishing section of TM. In other words, V is a smoothly-varying choice of nonzero tangent vector $V_p \neq 0$ in $T_p M$, the tangent space to M at point $p \in M$.

Suppose there are n linearly independent non-vanishing vector fields $V^1, V_2, \ldots, V^n$ on M. That is, for every point $p \in M$, the set of n vectors $\{V_p^1, V_p^2, \ldots, V_p^n\}$ forms a basis for $T_p M$.

Prove that TM is (isomorphic to) the trivial bundle $\mathbb{R}^n \times M$.

Hint: Build a vector bundle isomorphism between TM and $\mathbb{R}^n \times M$.

Your proof will hopefully also show that any rank-r bundle $\xi : E \to B$ with r linearly independent sections is the trivial bundle. I chose to frame this problem in terms of tangent vector fields for concreteness.