Floer Lasagna 2/18/26

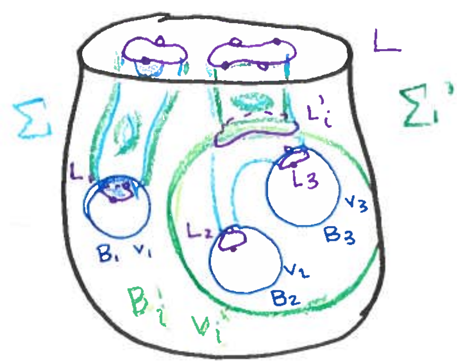
- W^4 = smooth, oriented
- $\mathbb{L} = \underbrace{(L, \vec{w}, \vec{z})}_{\text{possibly } \emptyset} \subseteq \underbrace{\partial W}_{\text{possibly } \emptyset}$

Chen: $\sigma(\text{"w"}) = U$, $\sigma(\text{"z"}) = V$

- Colorings are trivial: $\mathcal{R}_p^- = \mathbb{F}_2$

$\Downarrow$

- $\mathcal{CFL}^-(Y, \mathbb{L}^\sigma, s) = \widehat{CFL}(Y, \mathbb{L}, s)$



I. Constructions

Defn (Floer lasagna filling):

$$\mathcal{F} = (\underset{\text{input balls}}{B_i}, \underset{(L_i, \vec{w}_i, \vec{z}_i)}{\mathbb{L}_i}, \underset{\text{oriented, properly embdd}}{\Sigma}, \underset{\text{dividing arcs}}{\mathcal{A}}, \underset{\widehat{HFL}(\partial B_i, L_i)}{v_i})$$

s.t. $(W \setminus \cup \mathring{B}_i, \Sigma)$ is a decorated link cob. from $(\cup \partial B_i, \cup L_i)$ to $(\partial W, L)$

Aside: Σ can have closed components! But they must
- have ≥ 1 dividing arc
- have only closed dividing arcs
- split into Σ_w, Σ_z

Defn (Floer lasagna module):

$$\mathcal{FL}(W, \mathbb{L}) = \mathbb{F}_2\langle \text{fillings} \rangle / \text{ball removing}^*$$

* $(\Sigma, \mathcal{A})$ compatible w/ $(\Sigma', \mathcal{A}')$ as decorated link cobs

* $(\Sigma, \mathcal{A})\big|_{B_i \setminus \cup B_j}$ induces $F: \widehat{HFL}(\cup \partial B_j, \cup L_j) \to \widehat{HFL}(\partial B_i', L_i')$

s.t. $F(\otimes v_j) = v_i'$

Gradings: (see appendix 1 for well-definedness)

Disambiguation: Two conventions for absolute Maslov grading on $\widehat{HFL}$ related by

fixes top-degree gen. $x \in \widehat{HF}(S^3, \vec{w})$ to have $M(x) = 0$ → $$\underbrace{M(x)}_{\text{Chen}} = \underbrace{gr_w(x)}_{\text{Zemke}} - \frac{1}{2}(|\vec{w}| - 1)$$

"shift by Euler char."

$$M(\mathcal{F}) = \chi(\Sigma_w) + \sum_i M(V_i)$$

$$A(\mathcal{F}) = \frac{\chi(\Sigma_w) - \chi(\Sigma_z)}{2} + \sum_i A(V_i)$$

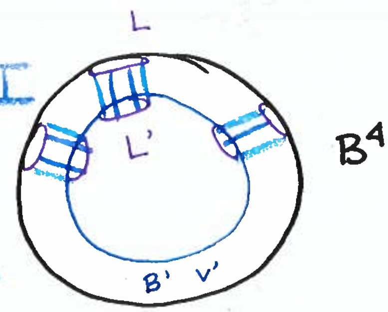
← sum of Alex. multigrading

Bonus: Also graded by $H_2(W, L; \mathbb{Z})$! (see appendix 1)

II. First calculations

Ex: $W = B^4$, $\mathbb{L} \subseteq \partial B^4 \cong S^3$

$\ell' = C \times I$



$\Sigma' = L \times I$

Obs: Every class $[\mathcal{F}]$ has a rep. $\mathcal{F}'$ that looks like ⟶ by eating everything w/ one ball

↳ $\Phi : \mathcal{FL}(B^4, \mathbb{L}) \xrightarrow{\simeq} \widehat{HFL}(S^3, L)$

$[\mathcal{F}] \longmapsto V'$

☆ $\mathcal{FL}$ extends $\widehat{HFL}$

$\Phi^{-1} : \widehat{HFL}(S^3, L) \xrightarrow{\simeq} \mathcal{FL}(B^4, \mathbb{L})$

$V \longmapsto [\mathcal{F}']$

constructed as above w/ input V

Ex: 4h attachment does nothing (see appendix 2)

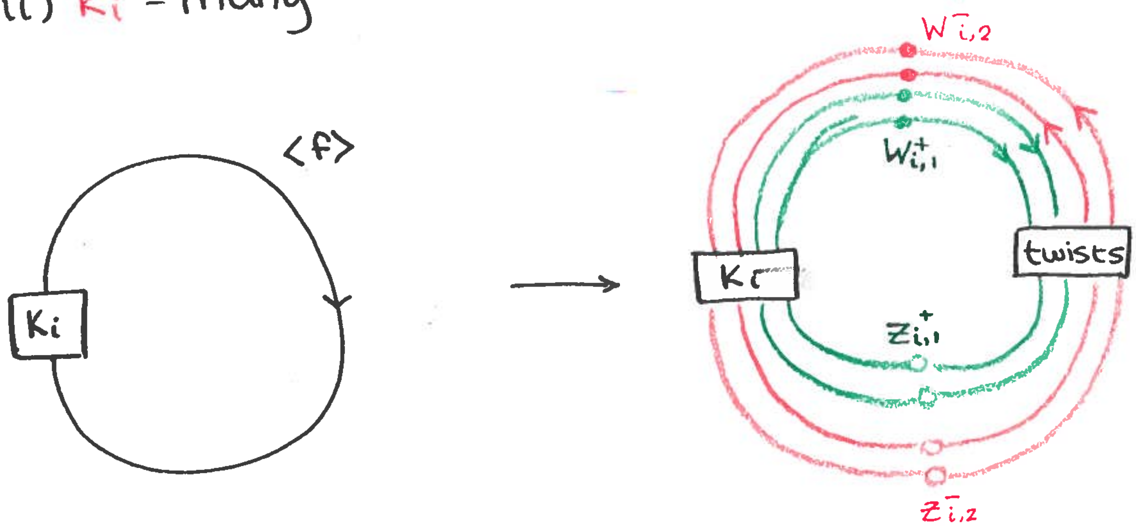
III. $c\widehat{HFL}$

Thm: Let W a 2-handlebody given by a framed link K and $\mathbb{L}$ in the 0-handle. Then $\exists$ a grading-preserving iso.

$$c\widehat{HFL}(\mathbb{L}, K) \cong \mathcal{FL}(W, \mathbb{L})$$

Setup:

- $K = K_1 \cup \cdots \cup K_n$ oriented, framed link in S^3
- $\vec{K}^+, \vec{K}^- \in \mathbb{N}^n$
- $K(\vec{K}^+, \vec{K}^-)$ = replace each K_i w/
  - (i) K_i^+-many parallel framings w/ $(+)$ orientation
  - (ii) K_i^--many " " " $(-)$ "



- $\widehat{HFL}(\mathbb{L}, K, \vec{K}^+, \vec{K}^-) := \widehat{HFL}(S^3, \mathbb{L} \cup K(\vec{K}^+, \vec{K}^-))$

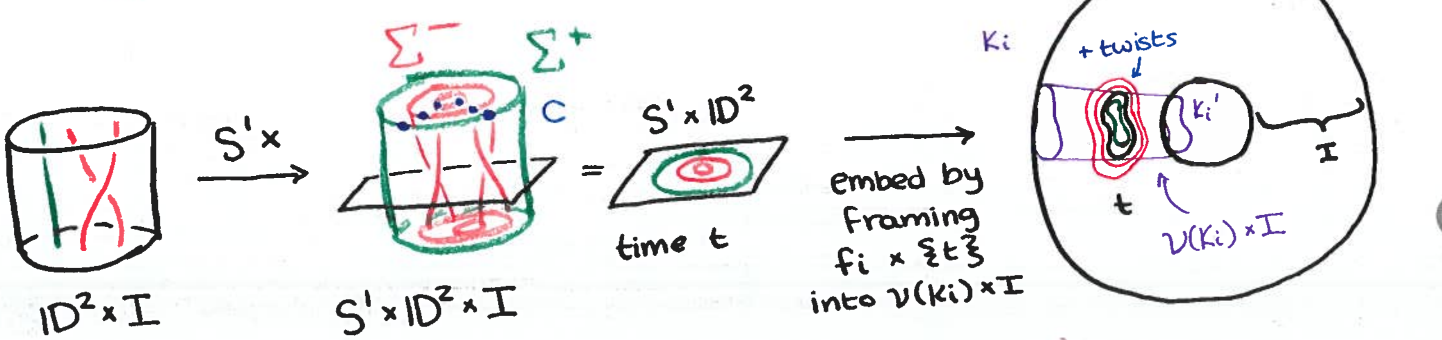
Defn:

$$\widehat{cHFL}(\mathbb{L}, K) := \bigoplus_{\vec{K}^+, \vec{K}^- \in \mathbb{N}^n} \widehat{HFL}(\mathbb{L}, K, \vec{K}^+, \vec{K}^-)\Big[\underbrace{\sum_{i=1}^n K_i^+ + \sum_{i=1}^n K_i^-}_{\text{shift in Maslov grading}}\Big] \Big/ \sim$$

Relation 1 (Braid gp action):

$$B_{K_i^-, K_i^+} = \left\{ \begin{array}{l} \text{Braids permuting} \\ \text{the } K_i^- \text{ strands} \end{array} \right\} \oplus \left\{ \begin{array}{l} \text{Braids permuting} \\ \text{the } K_i^+ \text{ strands} \end{array} \right\}$$

$\tau \in B_{K_i^-, K_i^+}$ determines a decorated link cob by



$\Sigma = (f_i \times id)(\Sigma^+ \cup \Sigma^-)$ $\cup$ cylinders on $K_{j \neq i}$ and L

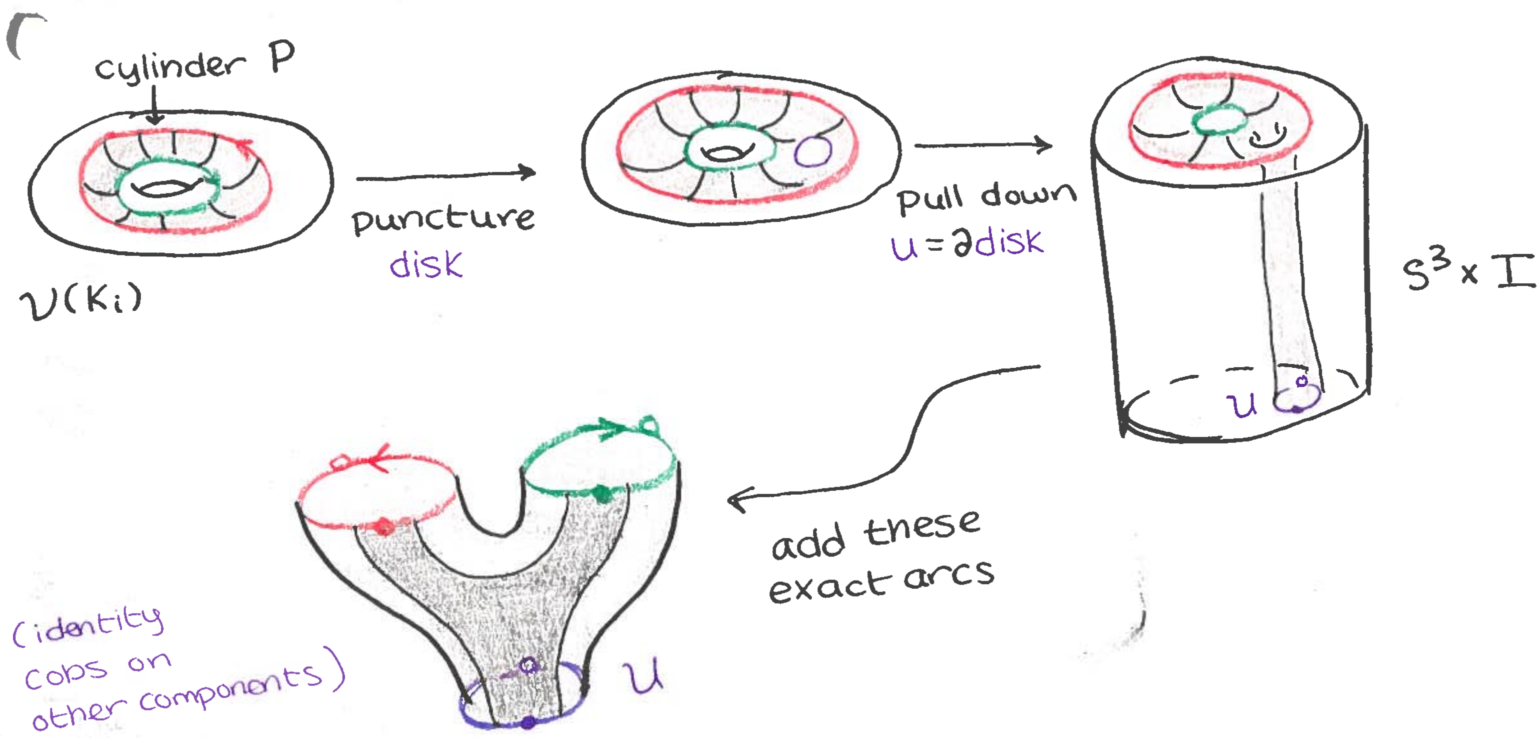
$\mathcal{A} = (f_i \times id)(C \times I)$, $\cup$ trivial divides on cylinders

which induces

$$F_\tau : \widehat{HFL}(\mathbb{L}, K, \vec{K}^+, \vec{K}^-) \to \widehat{HFL}(\mathbb{L}, K, \vec{K}^+, \vec{K}^-).$$

$$\Longrightarrow \boxed{v \sim F_\tau(v)}$$

Relation 2 (Pants):



$$F_P : \widehat{HFL}(\mathbb{L} \cup \mathbb{U}, K, \vec{K}^+, \vec{K}^-) \cong \widehat{HFL}(\mathbb{L}, K, \vec{K}^+, \vec{K}^-) \otimes \underbrace{\mathbb{F}_2\langle T, B\rangle}_{V}$$

$$\downarrow$$

$$\widehat{HFL}(\mathbb{L}, K, \vec{K}^+ + \underline{e_i}, \vec{K}^- + \underline{e_i})$$

add one more positive copy
and negative copy of framing

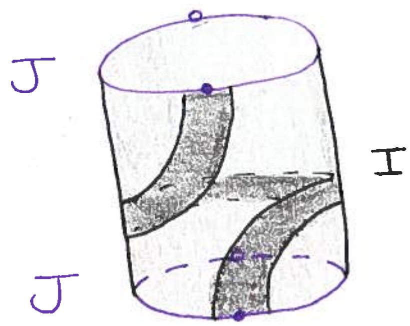
$$\Longrightarrow \boxed{\begin{array}{l} V \sim F_P(V \otimes B) \\ 0 \sim F_P(V \otimes T) \end{array}}$$

Relation 3 (Basepoint moving):

Let $j \in \{1, \dots, k_i^- + k_i^+\}$

$J = j^{th}$ parallel copy of K_i

The cobordism

identity cobs on other components $\cup$  $\subseteq S^3 \times I$

induces

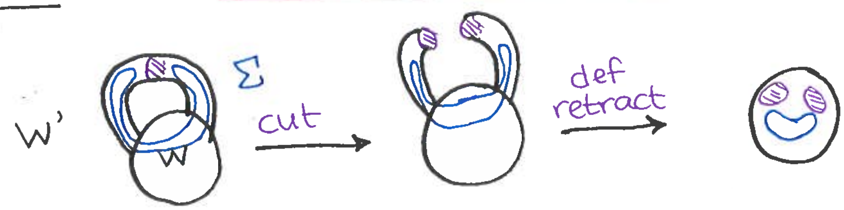
$$F_{m_{i,j}} : \widehat{HFL}(\mathbb{L}, K, \vec{k}^+, \vec{k}^-) \longrightarrow \widehat{HFL}(\mathbb{L}, K, \vec{k}^+, \vec{k}^-)$$

$$\Longrightarrow \boxed{v \sim F_{m_{i,j}}(v)}$$

Ex (3 & 4-handles):

Let $\mathbb{L} = \phi$, W cpct, $W' = W \cup K$-handle

Obs: If $\Sigma \cap \text{cocore} \overset{(*)}{=} \phi$, then



isotopes Σ into W.

By $\pitchfork$, $(*)$ can occur

$$\Longleftrightarrow 4 > \dim(\Sigma) + \dim(\text{cocore})$$

$$\Longleftrightarrow 4 > 2 + (4 - K)$$

$$\Longleftrightarrow K > 2$$

Hence for $K = 3, 4$ and $i : W \hookrightarrow W'$,

$$i_* : \mathcal{FL}(W, \phi) \longrightarrow \mathcal{FL}(W', \phi)$$
$$[\mathcal{F}] \longmapsto [i(\mathcal{F})]$$

is surjective.

When $K = 4$, $\dim(\Sigma \times I) + \dim(\text{cocore}) = 3 + 0 < 4$,

so $(*)$ holds for any isotopy $\varphi : \Sigma \times I \longrightarrow W'$.

Thus, $i_*([\mathcal{F}_1]) = i_*([\mathcal{F}_2])$ isotopic on handle

$\Rightarrow [\mathcal{F}_1] = [\mathcal{F}_2]$ isotopic off handle

$\Rightarrow i_*$ injective

$$\Rightarrow \boxed{\mathcal{FL}(W, \phi) \cong \mathcal{FL}(W', \phi)}$$

Appendix 1

Well-definedness of the gradings follows from

Thm 1.4 [Zem19b]: Let $(W,\mathcal{F}) : (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2)$ a decorated link cob. If $c_1(\mathfrak{s}|_{Y_1})$ and $c_1(\mathfrak{s}|_{Y_2})$ are torsion, then

$$gr_w(F_{W,\mathcal{F},\mathfrak{s}}(x)) - gr_w(x) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \chi(\Sigma_w) - \frac{1}{2}(|\vec{w}_1| + |\vec{w}_2|)$$

In particular,

(1) For $(W' = B_i' \setminus \cup \mathring{B}_j, \Sigma^* \cap W')$, $\mathrm{rk}\, H_n(W') = \begin{cases} 0 & n = 1,2,4 \\ 1 & n = 0 \\ |\mathbb{J}| & n = 3 \end{cases}$

$$\Rightarrow c_1(\mathfrak{s}) = \sigma(W') = 0$$
$$\chi(W') = |\mathbb{J}| - 1$$

Fact: $\Sigma_1 \cap \Sigma_2 = \{\text{arcs in } \partial\text{'s}\}$
$\Rightarrow \chi(\Sigma_1 \cup \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - \#\text{arcs}$

(2) $\chi(\Sigma_w) = \chi(\Sigma_w \cap W') + \chi(\Sigma_w^i) - |w_i|$

(3) Convert conventions $gr_w \longrightarrow M$

(4) Win: $M(\mathcal{F}') - M(\mathcal{F}) = M(V_i') + \chi(\Sigma_w^i) - M(\otimes V_j) - \chi(\Sigma_w) = 0.$

$\hookrightarrow$ A-grading also handled by Thm 1.4 in Zem19b

Relative homology classes

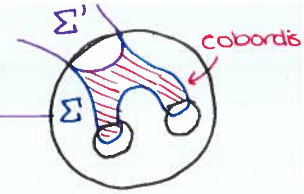
For a filling $\mathcal{F} = (B_i, \mathbb{L}_i, \Sigma, \mathcal{A}, V_i)$

$$[\Sigma] \in H_2(W, L \cup (\cup L_i)) \cong H_2(W, L)$$

$\longmapsto$ cap off L_i ∂-components w/ Seif. surf. in ∂B_i

remove caps $\longleftarrow$

$\mathcal{F} \sim \mathcal{F}' \Rightarrow [\Sigma] = [\Sigma']$. Hence



$$\mathcal{FL}(W, \mathbb{L}) = \bigoplus_{\alpha \in H_2(W, L; \mathbb{Z})} \underline{\mathcal{FL}(W, \mathbb{L}, \alpha)}$$

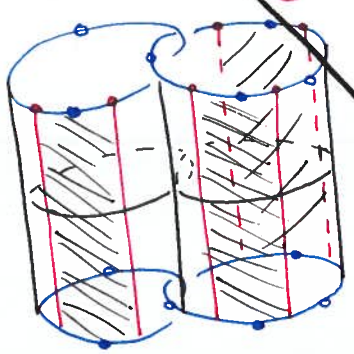
submodule gen'd by fillings s.t. $[\Sigma] = \alpha$

II. First Calculations

<u>Ex</u>: $W = B^4$. $\mathbb{L} = (L, \vec{w}, \vec{z}) \in S^3$

Every class $[\mathcal{F}]$ has a representative $\mathcal{F}'$ having one input ball $B' = W \setminus \nu(\partial W)$ s.t.

(1)



$\mathcal{A}' = C \times I$

$\Sigma' = L \times I$

(2) $(W \setminus B', \Sigma', \mathcal{A}')$ is the identity cobordism

This defines $\Psi : \mathcal{FL}(B^4, \mathbb{L}) \to \widehat{HFL}(S^3, L)$

$[\mathcal{F}] \mapsto v'$

w/ inverse $\Phi : \widehat{HFL}(S^3, L) \to \mathcal{FL}(B^4, \mathbb{L})$

$[v] \mapsto [\mathcal{F}']$ constructed as above

☆ $\mathcal{FL}$ extends HFL