LASAGNA + TQFTS (Part 2)

changed order of talks — move motivation and context first.

(I) Idea of Morrison - Walker - Wedrich's lasagna construction

Given: A TQFT

$$Z : \left\{ \begin{matrix} \text{framed links in } S^3, \\ \text{framed cobordisms in} \\ S^3 \times I \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{(bigraded)} \\ \text{modules} \\ \text{over } R \end{matrix} \right\}$$

where $Z(L \amalg L') = Z(L) \otimes Z(L')$

e.g. homology of c.h. type of chain cpx assoc to the link, KhR_2, KhR_N, $\widehat{HFL}$ (char)

Build: Skein lasagna module

W^4 = smooth oriented 4-mfld,

L = link in ∂W^4 (some 3-mfld, not nec. S^3)

as $\dfrac{R\langle \text{generators} = \text{“lasagna fillings” or “skeins”} \rangle}{\text{relations} \sim}$.

Rmk (1+1)-TQFT : "links" $\to$ R-mod (not quite right)

(3+1)-TQFT (relative) : pairs$(Y^3, L) \to$ R-mod.

(# Draw pic of filling in next relation section)

generators: $\mathcal{F} = (\Sigma, \{B_i, L_i, v_i\}_{i=1}^n)$

- $B_i = B^4 \subset \mathring{W}$
- Σ = surface (framed) properly embedded in $W - \coprod_i \mathring{B}_i$, $\partial\Sigma = L \cup \bigcup_i L_i$

(• L_i = link in $\partial B_i \cong S^3$, where Σ intersects ∂B_i)

- $v_i \in Z(L_i)$, so $v_1 \otimes \cdots \otimes v_n \in \bigotimes_i Z(L_i)$

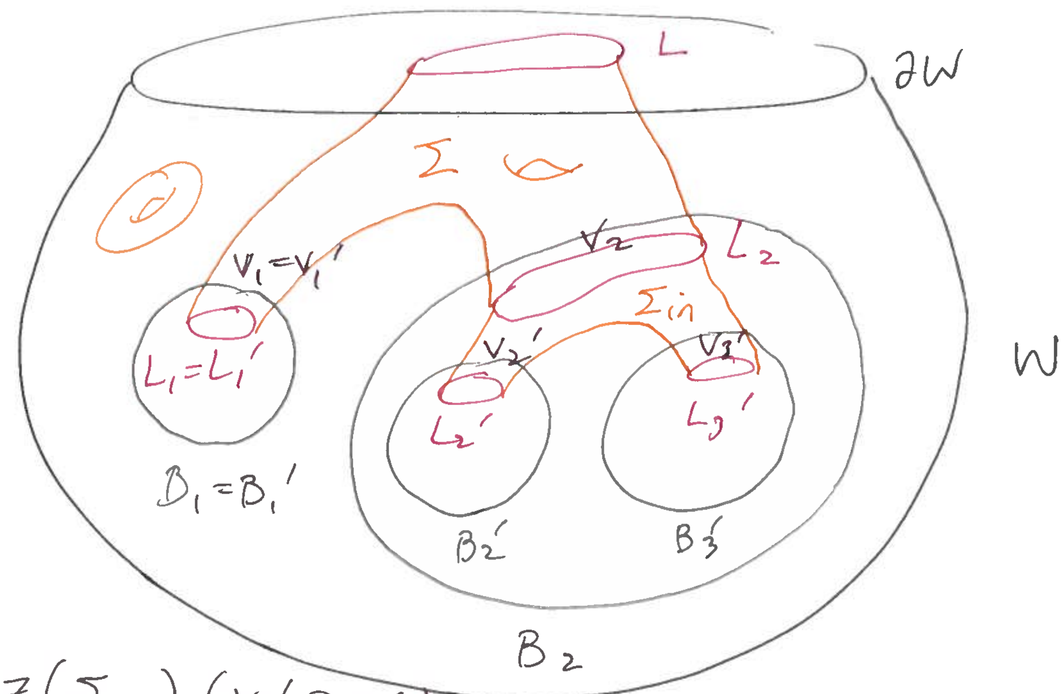
relations: "local", i.e. inside a ball

$\sim$ is linearly + transitively gen'd by

① isotopy of Σ rel ∂
(most data is at $\partial\Sigma$)

② enclosement

$\Sigma' = \Sigma \cup \Sigma_{in}$



$Z(\Sigma_{in})(v_2' \otimes v_3') = v_2$

$\mathcal{F} = (\Sigma, \{B_i, L_i, v_i\}) \sim \mathcal{F}' = (\Sigma', \{B_i', L_i', v_i'\})$

rmk. <u>Chen</u> $Z = \widehat{HFK}$; but this theory can handle links in arbitrary 3-mflds, and cobords in 4mflds (truly (3+1) rel. TQFT)

So we have some "less local" relations we can describe.

Chen's main result is adaptation of following to the HFK case:

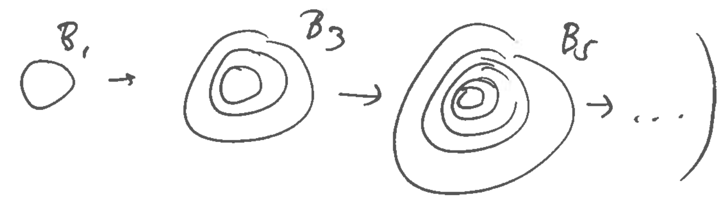
<u>Cabled Kh idea</u> (Manolescu – Neithalath)

If $W^4 = B^4 \cup \{$2-handles attached along framed link $K = K_1 \cup \dots \cup K_p\}$

can define <u>cabled Kh–Roz homology</u>

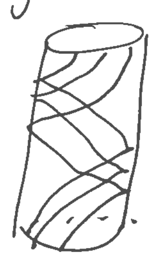
$$\underbrace{\underline{KhR_2}(W;L)}_{\text{computed using cables + some symmetrizing}} \cong \underbrace{S_0^{KhR_2}}_{\text{skein lasagna module}}(W;L)$$

eg. 

$\rightsquigarrow$ colim (

)

$\Rightarrow$ we will also need to keep an eye out for cabling in HFK:



(II) Ian's Restaurant

"you can get anything you want..."

<u>Goal</u> R char 2. (more specificity later)

$$\left\{ \begin{matrix} (Y^3, L) \\ (Y_1^3, L_1) \to (Y_2^3, L_2) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{comb data used} \\ \text{in HF homs} \\ (Y, \mathbb{L}, \mathfrak{s}) \ldots \end{matrix} \right\} \to R\text{-mod.}$$

{Domain category} (target cat)

- <u>ground rings</u> $\vec{P} = \{p_1, \ldots, p_n\}$. Define

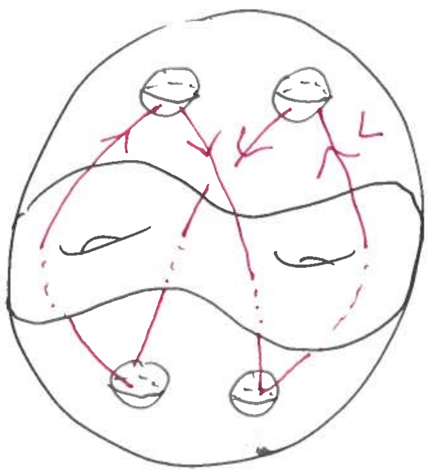
* Also $R^{\infty}_{\vec{p}}$

$$R_{\vec{p}} := \mathbb{F}_2[X_{p_1}, \ldots, X_{p_n}]$$

* may use w/taken = local.

For us: $\mathbb{F}_2[U_{\vec{w}}, V_{\vec{z}}] := \mathbb{F}_2[U_{w_1}, \ldots, U_{w_n}, V_{z_1}, \ldots, V_{z_n}]$

Dom Obj
- <u>links are encoded by basepoints</u> in the Heegaard diagram. Recall:



- $\Rightarrow$ we need at least 1 pair of w, z basepoints for each cpnt of L but we can have more pairs.
- 1 pair = $\uparrow$ and $\downarrow$.

- write $\mathbb{L} = (L, \vec{w}, \vec{z})$ viewing $\vec{w}, \vec{z}$ as points physically on the link.

- may sort them (pts) into colors, eg "w-basepoints" and "z-basepoints" via a "<u>coloring</u>"

$$\sigma : \vec{w} \cup \vec{z} \longrightarrow P$$

Det how you evaluate — which ring $R_{\vec{p}}$

- $\mathbb{L}^{\sigma} = (\mathbb{L}, \sigma) =$ "<u>multibased link</u>", the objects we care about

Ter obj, general

- <u>the invt associated to a multibased link</u>

$CFL^-(Y, \mathbb{L}^\sigma, \mathfrak{s})$ is a <u>curved chain cpx</u>

- chain gps are free, f.g. $\mathcal{R}_{\vec{p}}$-modules w/ $\mathbb{Z}^p$ Alex grading (filtration)
- There is a distinguished endomorphism ∂ s.t. $\partial^2 = \omega_{\mathbb{L},\sigma} \cdot id$ where $\omega_{\mathbb{L},\sigma} \in \mathcal{R}_{\vec{p}}$ is the "curvature".

↳ when $\omega_{\mathbb{L},\sigma} = 0$, we have an honest chain cpx.

<u>rmk later</u>

The invt is the filtered (by Alex filt. grading) chain htpy type.

Dom Mor.

- <u>Decorated link cobordisms</u> are the morphisms in the domain category.

Since our objects are $\mathbb{L}^\sigma = (L, \vec{w}, \vec{z}, \sigma)$, our morphisms need to keep track of each of these data.

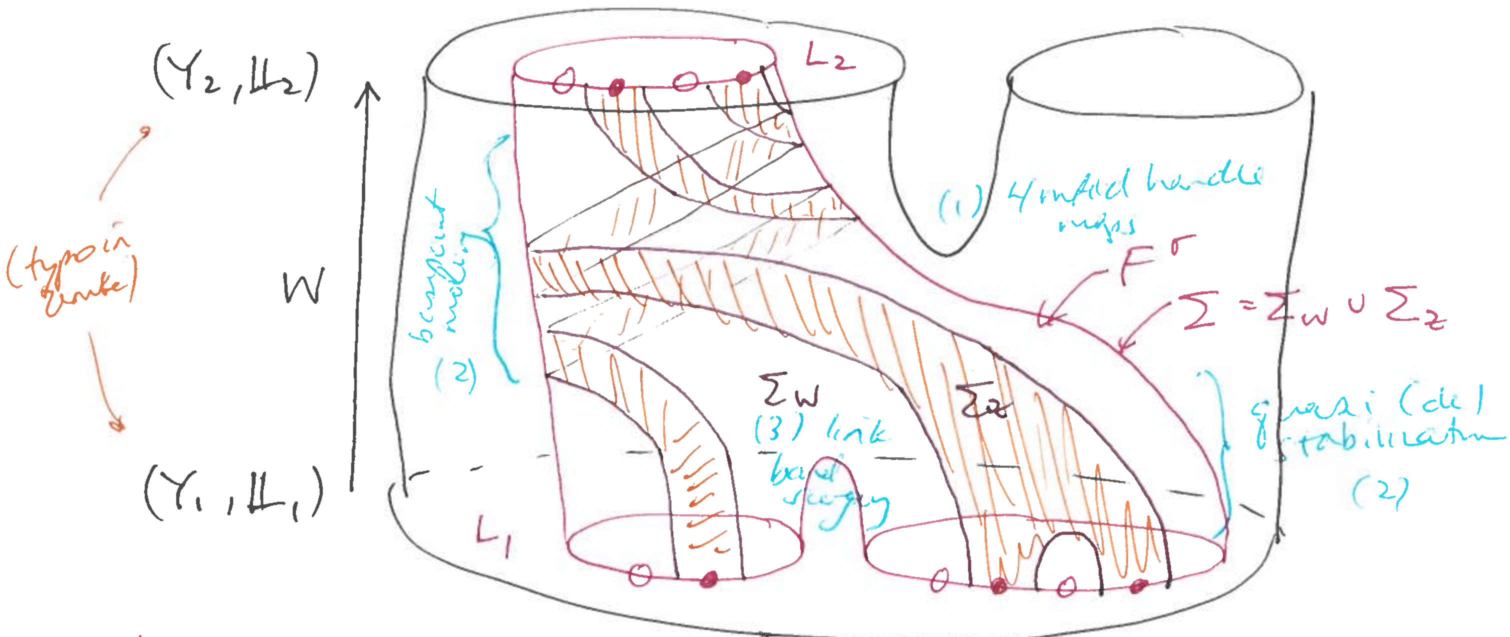
<u>defn</u> A <u>dec. link cob.</u> b/w two multibased links is

$$(W, F_\Sigma^\sigma) : (Y_1, \mathbb{L}_1) \longrightarrow (Y_2, \mathbb{L}_2)$$

extends σ_1 and σ_2 $\vec{w}^{(1)}, \vec{z}^{(1)}$ $\vec{w}^{(2)}, \vec{z}^{(2)}$

where

- W is a cpt 4-mfld, $\partial W = -Y_1 \cup Y_2$
- F^σ ($\mathcal{F}^\sigma$ in [Zemke]) is a colored, oriented surface w/ <u>divides</u>:



" $\circ = w$ "
$\bullet = z$

$\mathcal{A}$ = 'dividing curves' or "divides"
= where Σ_w and Σ_z meet.

The <u>coloring</u> of the surface F is

$$\sigma : \pi_0(\Sigma \setminus \mathcal{A}) \longrightarrow \vec{P}$$

where $\vec{w}^{(1)}_z \cup \vec{w}^{(2)} \subset \Sigma_w$,
$\vec{z}^{(1)} \cup \vec{z}^{(2)} \subset \Sigma_z$

↳ see how this "restricts" basically to the σ we had for $\mathbb{L}_1, \mathbb{L}_2$ at least up to order preserving sliding ...

for mor • The target morphisms are details we can discuss next time to some degree.

<u>Rmk.</u> Actually, to define the modules $CFL(Y, L^{\sigma}, \mathfrak{s})$, we have to make auxiliary choices: Heegaard diagram (Heegaard moves relate them) + ACS.

For invariance under these we actually need to associate a whole <u>transitive system</u>

<u>defn.</u> $\mathcal{C}$ = category, A = diagram category ("shape")

for us, (curved) cpxs over R, up to filtered, equivariant ch.e

set of all $\mathfrak{s}$-admissible Heegaard diagrams for $(Y^3; L)$

A <u>transitive system over $\mathcal{C}$ indexed by the diagram cat A</u> is a functor

$$C : A \to \mathcal{C} \quad \text{w/ distinguished morphism}$$

$$\Phi_{a \to a'} : C(a) \to C(a').$$

A <u>morphism of transitive systems</u> is a collection of morphisms F_{a_1, a_2} s.t.

$$\begin{tikzcd}
C_1(a_1) \arrow[r, "F_{a_1,a_2}"] \arrow[d, "\Phi_{a_1 \to a_1'}"'] \arrow[dr, phantom, "\circlearrowright"] & C_2(a_2) \arrow[d, "\Phi_{a_2 \to a_2'}"] \\
\underbrace{C_1(a_1')}_{(Y_1^3, L_1)} \arrow[r, "F_{a_1', a_2'}"'] & \underbrace{C_2(a_2')}_{(Y_2, L_2)}
\end{tikzcd}$$