

Goal/Overview

Let X be a top space. We wish to develop an 'invariant' of X :

Combinatorial description
of X



$\mathbb{Z}$ -graded $\mathbb{Z}$ -module

$\uparrow$ one $\mathbb{Z}$ -module per dimension
 $d \geq 0$, actually

$\nearrow$ i.e. well-defined
assignment to X

called the "homology of X ".

We will discuss multiple (equivalent) definitions of "the homology":

① simplicial homology

- computed from a Δ -complex structure on X

② singular homology

- less straight forward / combinatorial but more robust

Sometimes properties are easier to prove when the construction is more robust...

③ CW homology

- computed from a CW-complex structure on X

- most computationally efficient, most practical
and the one you really ought to learn to compute!

Simplicies plural of "simplex"

basic building blocks for (some) spaces

defn The n -simplex Δ^n is the convex hull of

the $n+1$ points $\{v_0, \dots, v_n\} = \{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$

- conv. to the standard basis vectors $e_1, \dots, e_n$
- v_i for "vertices"

$$\Delta^n := \left\{ \underbrace{(t_0, \dots, t_n)} \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0 \forall i \right\}$$

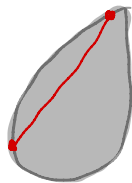
$= \sum_{i=0}^n t_i v_i$ since our v_i are standard!

Aside The convex hull of a set of points $P = \{p_1, \dots, p_k\}$ in $\mathbb{R}^N$ is the smallest convex set containing all points in P

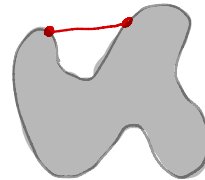
↖ For any $x, y \in S$, all points on the line b/w x and y , are also in S

Convex:

Some examples
in $\mathbb{R}^2$:



Not convex:



This is equivalent to the set of all linear combinations of points $p_i \in P$ where the nonnegative weights t_i add up to 100%:

$$\text{Hull}(P) = \left\{ \sum_{i=1}^k t_i p_i \mid \sum_{i=1}^k t_i = 1, t_i \geq 0 \forall i \right\}$$

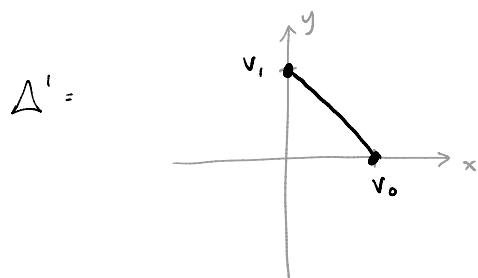
eg. the first few simplices:

(geometrically)

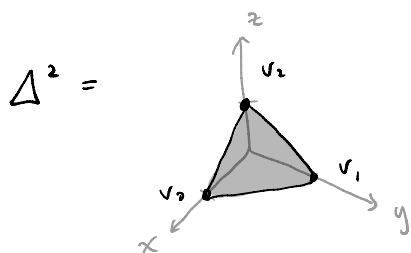


(abstractly / combinatorially)

the ordered set
 $[0] = \{0\}$. actually (0)
 b/c ordered.



the ordered set
 $[1] = \{0, 1\}$ is $(0, 1)$

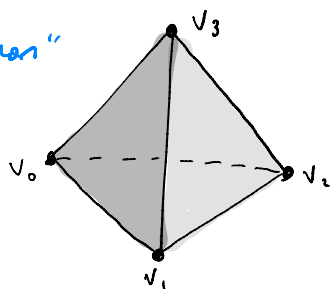


the ordered set
 $[2] = \{0, 1, 2\}$ is $(0, 1, 2)$

$\Delta^3 = \dots$ well it looks like this, but embedded in $\mathbb{R}^4$:

the ordered set $[3]$

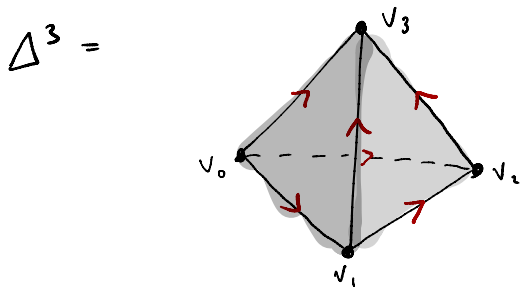
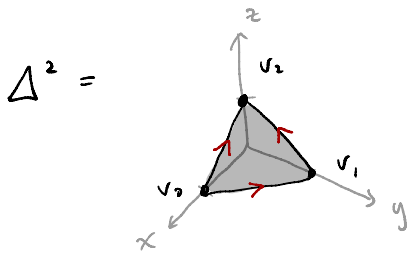
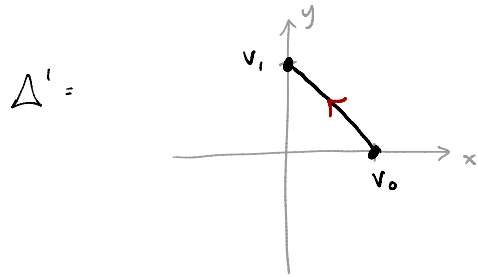
"tetrahedron"



Remark. Δ^n is an n -dim manifold, ie for any point in the interior, a neighborhood looks like $\mathbb{R}^n$.

The ordering of vertices in a simplex will be very important.

In particular, the order gives us an orientation on all edges:



notation we capture all the abstract data of the n -simplex by the ordered set

$$[n] := \{0, 1, 2, \dots, n\}$$

maybe one likes to write $(0, 1, \dots, n)$ but I'm not thinking of these as coordinates...

with the total ordering $\leq$.

⚠ Some notation warnings:

- In CS, $[n]$ usually means $\{1, 2, \dots, n\}$
- In homological algebra, $[n]$ means degree shift of n
- In quantum algebra, $[n]$ is a quantum integer.

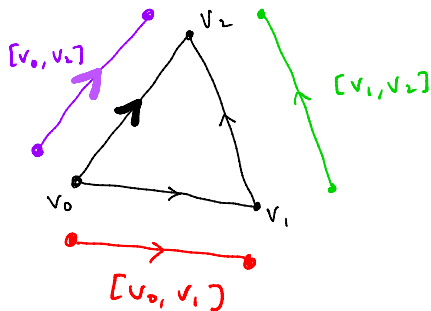
defn A face (really, "subsimplex") of a simplex Δ^n

is the subsimplex with vertices a nonempty subset of $\{v_0, \dots, v_n\}$.

- A k -dim face of Δ^n (where $0 \leq k \leq n$) is a subsimplex homomorphic to Δ^k
- Hatcher's notation: the subsimplex with vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ with $i_1 < i_2 < \dots < i_k$ is written $[v_{i_1}, v_{i_2}, \dots, v_{i_k}]$
- A k -dim face of Δ^n corresponds to an order-preserving map $[k] \longrightarrow [n]$.

eg. Δ^2 has 3 codimension-1 (ie dimension 1) faces:

If X is n dim'l,
and $S \subset X$ is d dim'l,
the codim of S is $n-d$.



These can be written using "delete" notation:

- $[v_1, v_2] = [\hat{v}_0, v_1, v_2]$
- $[v_0, v_2] = [v_0, \hat{v}_1, v_2]$
- $[v_0, v_1] = [v_0, v_1, \hat{v}_2]$

These also correspond to the 3 order-preserving maps

$[1] \rightarrow [2]$:

- $[v_1, v_2] = [\hat{v}_0, v_1, v_2] : 0 \mapsto 1, 1 \mapsto 2$
- $[v_0, v_2] = [v_0, \hat{v}_1, v_2] : 0 \mapsto 0, 1 \mapsto 2$
- $[v_0, v_1] = [v_0, v_1, \hat{v}_2] : 0 \mapsto 0, 1 \mapsto 1$

ex. "ex" = exercise in my notes; "eg" = example

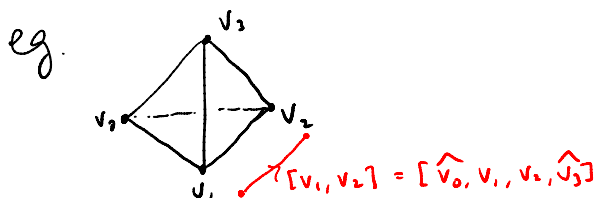
(a) How many faces of dimension k are there in Δ^n ?

$\binom{n+1}{k+1}$ faces. You keep $k+1$ vertices.

(b) How many faces does Δ^n have total?

nonempty subsets of $[n] = 2^{n+1} - 1$.

ex. Identify and draw all faces of Δ^3 , w/ orientation for dim 1 and 2 faces.



Δ -complexes

(soft intro)

Informal defn

A Δ -complex is a quotient space of a collection of disjoint simplices obtained by identifying faces in the order-preserving way

eg $S^1 = \text{the circle}$

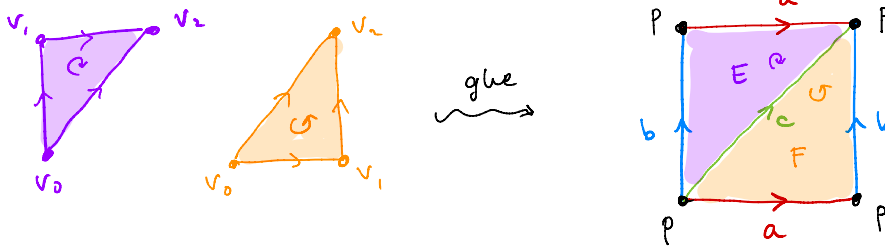


There are many different ways to build S^1 !

eg. $S^2 = \text{the sphere}$



eg. the torus $T = S^1 \times S^1$:



Here is the formal definition:

defn. Let X be a topological space.

A Δ -complex structure on X is a collection of maps

$$\{\sigma_\alpha: \Delta^{n(\alpha)} \rightarrow X\}_{\alpha \in A} \quad \text{such that}$$

$$\boxed{\Delta-1} \quad \forall \alpha \in A, \quad \sigma_\alpha|_{\text{int}(\Delta^{n(\alpha)})} \text{ is injective}$$

and each point in X is in the image of exactly one such restriction

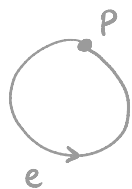
* Caveat For $S \subset \mathbb{R}^d$, the interior $\text{int}(S)$ of S is defined to be the union of all open subsets U of $\mathbb{R}^d$ contained in S :

$$\text{int}(S) = \bigcup \{U \subset \mathbb{R}^n \mid U \subseteq S\}.$$

Since Δ^0 is a single point in $\mathbb{R}^1$, $\text{int}(\Delta^0) = \emptyset \dots$

So, for the purposes of this definition, say " $\text{int}(\Delta^0) = \Delta^0$ ".

eg. $X = S^1$



point p is in the image of

$$\sigma_p: \Delta^0 \rightarrow X$$

but not in $\sigma_e|_{\text{int}(\Delta^1)}: \text{int}(\Delta^1) \rightarrow X$.

$\boxed{\Delta-2}$ Each restriction $\sigma_\alpha|_{\text{face } F \text{ of } \Delta^{n(\alpha)}}$ where $F \cong \Delta^k$

is equal to some map $\sigma_\beta: \Delta^k \rightarrow X$

↑

Here we need to identify $\Delta^{n(\beta)} = \Delta^k$, the src of σ_β with the face F of $\Delta^{n(\alpha)}$ in the order preserving way.

⚠ There are geometric concerns about parametrization.

We claim there is an "obvious" canonical way to identify F with Δ^k in the order-preserving way and speak no more of this.

$\boxed{\Delta-3}$ $U \subset X$ is open $\iff \forall \alpha \in A, \sigma_\alpha^{-1}(U)$ is open.

//

Rmk. (Casual interpretation of the requirements)

($\Delta-1$) tells us that X is built specifically and entirely from these pieces.

($\Delta-2$) tells us that if a simplex Δ^n is used to build X , then all its faces (subsimplices) are also part of the construction, i.e. they are domains of other maps in the collection.

($\Delta-3$) tells us that the quotient topology on the constructed space agrees with the topology of the originally given space X .

mk. (Aside)

- ① Cardinality of A need not be finite, but we will mostly work with finite data

ex. (Remark about cardinality of A). Put a Δ -complex structure on the Hedgehog space for cardinality K .

- ② Not all spaces have a Δ -structure!

Fact A closed 4-manifold is triangulable iff it is smoothable.

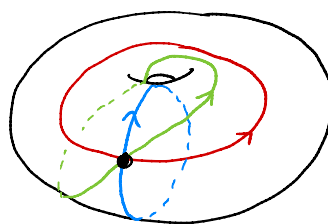
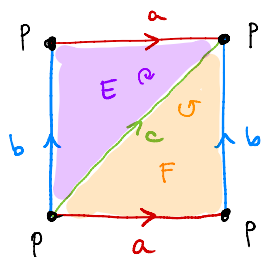
I have not provided a definition here, but it is equivalent to being the geom. realization of a simplicial set.

eg. The E^8 manifold (!! It's even a manifold!) is not triangulable.

- ③ However, Fact every manifold is homotopic (but not necessarily homeomorphic) to a CW complex.

Examples Keep these in your pocket!

eg. Thus $T = S' \times S'$ (more formally how)



The collection of maps: $|A| = 6$ maps

- $\sigma_p : \Delta^0 \rightarrow X$ sends $v_0 \mapsto p$
 - $\sigma_a : \Delta^1 \rightarrow X$ sends $[v_0, v_1]$ to a
 - $\sigma_b : \Delta^1 \rightarrow X$ sends $[v_0, v_1]$ to b
 - $\sigma_c : \Delta^1 \rightarrow X$ sends $[v_0, v_1]$ to c
- } via the "canonical" (i.e. obvious) homeomorphism

All three of $\sigma_a, \sigma_b, \sigma_c$ send the endpoints $v_0, v_1 \mapsto p$

- $\sigma_E : \Delta^2 \rightarrow X$ sends $[v_0, v_1, v_2] \rightarrow E$

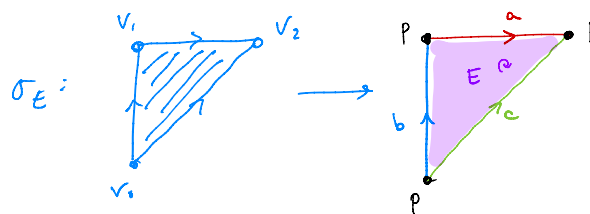
and in particular sends

$$[v_0, v_1] \rightarrow b$$

$$[v_1, v_2] \rightarrow a$$

$$[v_0, v_2] \rightarrow c$$

and sends $[v_0], [v_1], [v_2] \rightarrow p$.



- $\sigma_F : \Delta^2 \rightarrow X$ sends $[v_0, v_1, v_2] \rightarrow F$

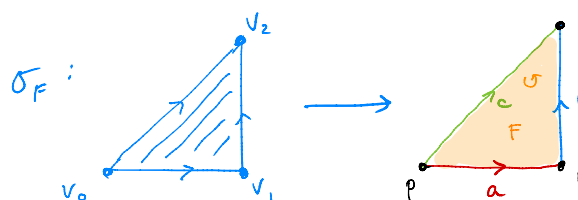
and in particular sends

$$[v_0, v_1] \rightarrow a$$

$$[v_1, v_2] \rightarrow b$$

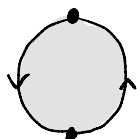
$$[v_0, v_2] \rightarrow c$$

and sends $[v_0], [v_1], [v_2] \rightarrow p$.



eg. $\mathbb{RP}^2$: 2-dim real projective space
 $= S^2 / \sim$ where $\sim$ identifies antipodal points

We often think of $\mathbb{RP}^2$ topologically as the space we get from identifying two sides of a bigon:

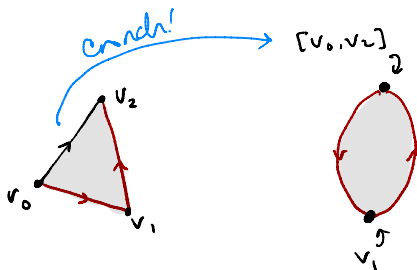


Δ NOT a duple.
 Note the orientations.

This is not a Δ -complex structure. Here's a thought process:

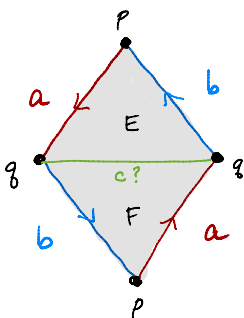
Attempt 1: Bad

Let's try a single Δ^2 :



Violates
 $\Delta-1$
 injectivity.

Attempt 2: better, but still not legal



Now the path ba is mapped to the "seam"

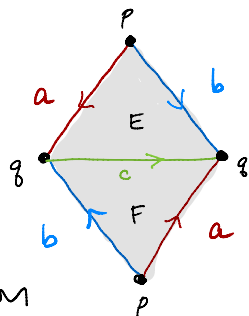
There is no way to orient c so that

$$\sigma_E|_{\text{face "c"}} = \sigma_F|_{\text{face "c"}}$$

Either choice violates $\Delta-2$.

Attempt 3 slight modification, how good

Just reverse the orientation of b :



Verify that now:

$$\sigma_E|_{[v_1, v_2]} = \sigma_c = \sigma_F|_{[v_0, v_1]}$$

We choose an orientation on c .

(Δ -)

ex. Build simplicial structures for

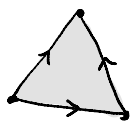
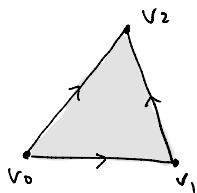
(a) spheres of all dims

(b) cco surfaces cco = closed, catd, orientable

(c) cc, non-orientable surfaces

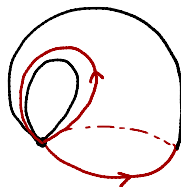
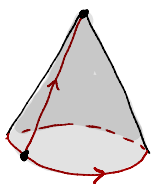
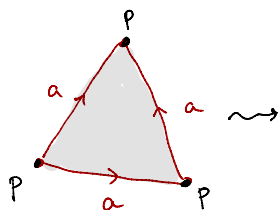
(d) n-torus

eg. "Dunce cap" what can I make with just one Δ^2 (+ all faces)?



i.e. identify all 3 sides, according to these orientations!

One can literally crochet/knit/darn this! (in $\mathbb{R}^3$!)



then identify the red circles...

ex. (maybe) Challenge / review for 215a:

Show that the dunce cap is $\cong D^2 \subseteq *$ (homotopy equiv to a point!).

Soft intro to simplicial homology chain complex

We wish to translate the Δ -cpx structure to algebra, in the form of a chain complex of abelian groups.

↳ later we'll see that the homology of this chain cpx does not depend on the choice of Δ -cpx structure.
But for now we just want to build a chain cpx.

Chain complex: 2 pieces of data:

- a $\mathbb{Z}$ -mod C_i for each dimension $i \in \mathbb{Z}$ "chain groups"
(most are 0, and for us $C_i = 0$ when $i < 0$).
- a "differential", ie maps $\partial_i: C_i \rightarrow C_{i-1} \quad \forall i$.
Such that $\partial^2 = 0$ ie $\partial_{i+1} \circ \partial_i = 0 \quad \forall i$.

In our case, $C_i = \mathbb{Z} \langle \{\sigma_\alpha\} \mid n(\alpha) = i \rangle$

and ∂_i records the image of $\partial \Delta^{n(\alpha)}$.

↳ also sometimes called "the boundary map" for this reason.

eg. $\Delta' = v_0 \rightarrow v_1$ is a space with a Δ cpx structure, where the σ_α are inclusions.

0-simplices: $\sigma_\alpha: \Delta^0 \rightarrow \Delta'$ $\sigma_\beta: \Delta^0 \rightarrow \Delta'$
 $v_0 \mapsto v_0$ $v_0 \mapsto v_1$

1-simplicies: $\sigma_\gamma: \Delta' \rightarrow \Delta' = \text{id}_{\Delta'}$

} Blue Δ' is thought of as the top space, ie $X = \Delta'$.

Chain complex let $X = \Delta'$

Here's what's actually happening:

chain groups: $C = \bigoplus_{i \in \mathbb{Z}} C_i$, $C_0 = \mathbb{Z}\langle \sigma_\alpha, \sigma_\beta \rangle$, $C_1 = \mathbb{Z}\langle \sigma_\gamma \rangle$,
 all other $C_i = 0$.

differential: $\partial_i: C_i \rightarrow C_0$ is determined by
 $\sigma_\gamma \mapsto \sigma_\beta - \sigma_\alpha$
 (all other $\partial_i = 0$)

How did I get this? We actually think about where the boundaries of $\Delta^{n(\alpha)}$ are sent!

We actually may identify σ_α with its image in this example

so we may write (as we did Wednesday):

$\partial[v_0, v_1] = [v_1] - [v_0]$
 ↑ Final initial.

I omitted the subscript bc I can think of

$\partial = \bigoplus_i \partial_i$ as a total differential on the
 module $C = \bigoplus C_i$

Since all other $\partial_i = 0$, it is clear that ∂ satisfies $\partial^2 = 0$.

↳ ex. If you've never seen this before, write $\partial = \partial_0 + \partial_1 + \dots$
 and write down the graded pieces of the equation
 to deduce that $\partial_{i+1} \circ \partial_i = 0$.

Quick start guide to chain complexes, homology

Let's start with some algebra. Then we'll describe simplicial homology formally with additional standard notation, terminology

- **Chain groups**: $C = \bigoplus C_i$
- **differential**: $\partial = \sum \partial_i$, where $\partial^2 = 0$
- **chain complex**: (C, ∂)
- **homology functor**: $\{\text{chain cpxs}\} \longrightarrow \{\text{graded } \mathbb{Z}\text{-mods}\}$

$$\text{by } H_i = \frac{\ker \partial_i}{\text{im } \partial_{i+1}}$$

↑ cycles
↑ boundaries

note This is only ok to write if you know $\text{im } \partial_{i+1} \subset \ker \partial_i$.
This is ensured by $\partial_{i+1} \circ \partial_i = 0$.

- Two cycles are **homologous** if they differ by a boundary. Then they represent the same **homology class**.
- The point of homology: associating different decompositions of the space X (diff Δ -structures) may yield different chain complexes, but these all compute the same homology groups $\{H_i(X)\}_{i \in \mathbb{Z}}$. (all together, we write $H_*(X)$).

* This point is very top-spaces-centric. In general:

$$\underbrace{\text{chain htpy equivalence}} \Rightarrow \underbrace{\text{quasi-isomorphism}}$$

stronger equivalence; analogous to "homotopy equiv" for topspaces

= have the same homology groups

- "**graded**" just means you associate integers (the dimension) to different pieces of (C, ∂)

Simplicial Homology

- let X be a Δ -complex.
- let $\Delta_n(X)$ be the free abelian group generated by the open n -simplices e_α^n of X .

- By $\Delta-1$, it's ok to think of these e_α^n as part of X , the target of the map $\sigma_\alpha: \Delta^n \rightarrow X$

We call σ_α the characteristic map of e_α^n .

- We write each element as finite formal sums

$$\sum_{\{\alpha \in A \mid n(\alpha) = n\}} c_\alpha \cdot \sigma_\alpha$$

↑ b/c we are working with $\mathbb{Z}\sigma_\alpha \oplus \mathbb{Z}\sigma_\beta \oplus \dots \leftarrow$ direct sums.

or we could write $\sum c_\alpha e_\alpha^n$, but I won't, in general...

- The boundary homomorphism $\partial_n: \Delta_n(X) \rightarrow \Delta_{n-1}(X)$ is given by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \big|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}$$

↖ basis elt.
↑ where $n(\alpha) = n$

by $\Delta-2$ we have this map as a basis element for $\Delta_{n-1}(X)$

Claim (lemma 2.1) let $\mathcal{C} = \bigoplus_n \Delta_n(X)$ and $\partial = \sum_n \partial_n$

Then $(\mathcal{C}, \partial)$ is a chain complex.

Pf.

It suffices to show that the composition

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Calculate directly from the definition:

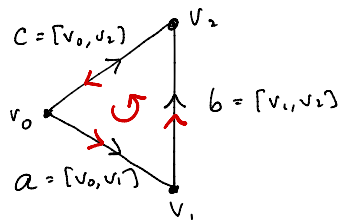
$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$\partial_{n-1} \partial_n(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}$$

note the
parity change
b/c v_i was
deleted!

opp signs.
These
cancel.

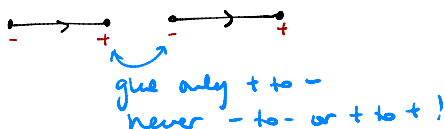
Remark What's happening topologically?



on a simplex, with orientations of codim 1

faces determined by the ∂ orientation of the interior (ie "cyclic", at least in the shown dimension),

all codim 2 faces are used twice, but w/ opp. sign (orientation)



Defn We call $H_*(\mathcal{C}, \partial)$ the simplicial homology of X

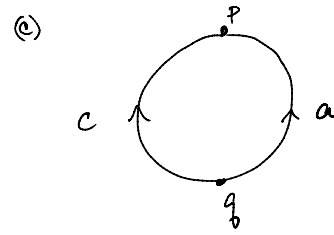
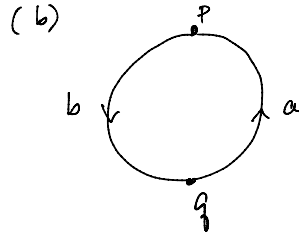
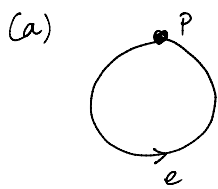
and write it as $H^*(X) = \bigoplus_n H_n(X)$

T 1-2; R 10-11? 11-12? Elli?

Examples: (Compare + Contrast)

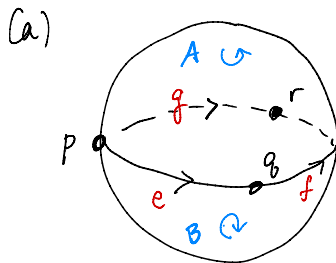
① S^1 , w/ diff't Δ -cpx structures

let X in all cases denote the space w/ the Δ -structure.

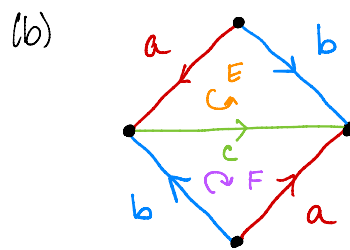


(check that these really are Δ -cpx structures...)

② S^2 vs $\mathbb{RP}^2$ check these are Δ -cpxs.

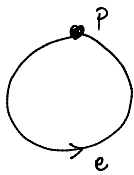


orientable



non orientable!

eg 1a



$$\Delta_0 = \mathbb{Z}\sigma_0 \text{ where } \sigma_0: \Delta^0 \rightarrow X, [v_0] \mapsto p$$

$$\Delta_1 = \mathbb{Z}\sigma_1 \text{ where } \sigma_1: \Delta^1 \rightarrow X, [v_0, v_1] \rightsquigarrow e$$

↑ homs

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

only ∂_1 is nontrivial, determined by how it maps basis element σ_1 :

$$\partial_1 \sigma_1 = (-1)^0 \sigma_1|_{[v_0]} + (-1)^1 \sigma_1|_{[v_1]} = \sigma_0 - \sigma_0 = 0.$$

So our chain complex is

$$0 \rightarrow C_1 = \mathbb{Z}\sigma_1 \xrightarrow{\quad} C_0 = \mathbb{Z}\sigma_0 \rightarrow 0$$

$$\sigma_1 \mapsto 0$$

More abstractly:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}} \rightarrow 0$$

↑ underline where C_0 is
("homological degree 0")

Take homology:

$$H_0^A(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\mathbb{Z}\sigma_0}{\{0\}} \cong \mathbb{Z}.$$

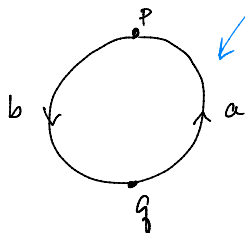
$$H_1^A(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\mathbb{Z}\sigma_1}{\{0\}} \cong \mathbb{Z}$$

Therefore the simplicial homology of X is

$$H_n^A(X) = \begin{cases} \mathbb{Z} & n=0,1 \\ 0 & \text{otherwise} \end{cases}$$

eg 1b

$X =$



notation:
 $\sigma_a: \Delta^1 \rightarrow X$

$$0 \rightarrow C_1 = \mathbb{Z}\langle a, b \rangle \rightarrow C_0 = \mathbb{Z}\langle p, q \rangle \rightarrow 0$$

$$a \mapsto p - q$$

$$b \mapsto q - p$$

actually:

$$\sigma_a \mapsto \sigma_p - \sigma_q$$

$$\sigma_b \mapsto \sigma_q - \sigma_p$$

* Abusing notation:
writing σ for σ_x .

- $\ker \partial_0 = \mathbb{Z}\langle p, q \rangle$

$$\text{im } \partial_1 = \mathbb{Z}\langle p - q \rangle$$

p is homologous to q

so they are in the same equiv. class in

homology (a subquotient group)

$$\Rightarrow H_0^A(X) = \mathbb{Z}\langle p, q \rangle / \langle p - q \rangle = \mathbb{Z}\langle [p] \rangle \cong \mathbb{Z}$$

$$[p] = [q]$$

- $\ker \partial_1 = \underline{???} = \mathbb{Z}\langle a + b \rangle$

$$\text{im } \partial_1 = 0$$

$$\Rightarrow H_1^A(X) = \mathbb{Z}\langle p + q \rangle \cong \mathbb{Z}$$

- Thus
$$H_i^A(X) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & \text{o/w} \end{cases}$$

* Same as eg 1a!

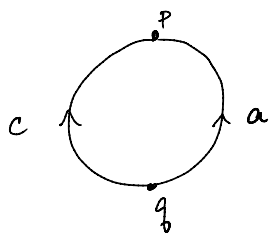
More definitions (Important)

- An element of some C_i is called a chain.
- An element of some $\ker \partial_i$ is called a cycle.
- An element of some $\text{im } \partial_i$ is called a boundary.
- Two cycles $x, y \in \ker \partial_i \subset C_i$ homologous if they differ by a boundary, i.e. $y - x \in \text{im } \partial_{i+1}$.

We will now write a bit less.

eg 1c

$X =$



$$\Delta_0 = \mathbb{Z}\langle p, q \rangle$$

$$\Delta_1 = \mathbb{Z}\langle a, c \rangle$$

$$\partial_1 a = p - q$$

$$\partial_1 c = p - q$$

$$H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{Z}\langle p, q \rangle / \langle p - q \rangle \cong \mathbb{Z}$$

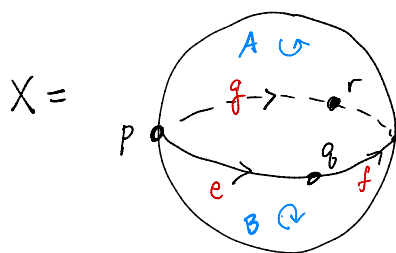
$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \mathbb{Z}\langle a - c \rangle / 0 \cong \mathbb{Z}$$

Same homology.

We've shown in 3 ways that $H_*(S^1)$ is

$\mathbb{Z}$ in dims 0 and 1, and 0 otherwise.

eg 2a

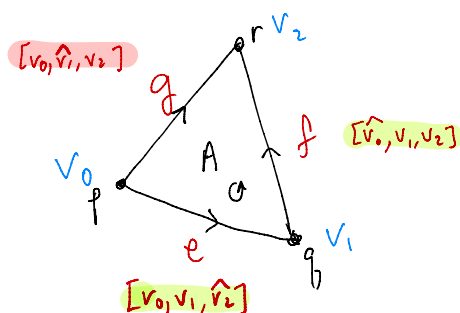


$$0 \rightarrow \mathbb{Z}\langle A, B \rangle \longrightarrow \mathbb{Z}\langle e, f, g \rangle \longrightarrow \mathbb{Z}\langle p, q, r \rangle \longrightarrow 0$$

• Boundary maps

$$\partial_1 e = q - p \quad \partial_1 f = r - q \quad \partial_1 g = r - p$$

$$\partial_2 A = e - g + f$$



* You can also get this by comparing ∂ orientation of ∂A and the simplicial orientation; see HW01 for Δ^3 example.

$$\text{Similarly } \partial B = e - g + f$$

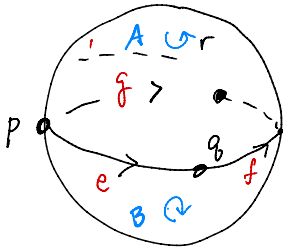
• Homology

$$H_0^A(X) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}\langle p, q, r \rangle / \langle \underbrace{q - p, r - q, r - p}_{\text{linear dependence!}} \rangle$$

change basis:

$$= \mathbb{Z}\langle p - q, q - r, r \rangle / \langle q - p, r - q \rangle \cong \mathbb{Z}\langle r \rangle \cong \mathbb{Z}.$$

TBC'd either Tuesday Disc. recorded or on HW on your own!



$$H_1^A(X) = \ker \partial_1 / \text{img } \partial_2 = \mathbb{Z} \langle e-g+f \rangle / \mathbb{Z} \langle e-g+f \rangle \cong 0.$$

this is non obvious!

take a moment to confirm

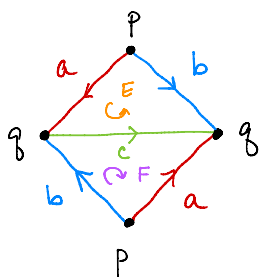
$$H_2^A(X) = \ker \partial_2 / \text{img } \partial_3 = \mathbb{Z} \langle A-B \rangle / 0 \cong \mathbb{Z}$$

• therefore

$$H_n^A(X) \cong \begin{cases} \mathbb{Z} & n=0,2 \\ 0 & \text{o/w.} \end{cases}$$

* compare w/ $H_*(S')$.

eg 2b



$$0 \rightarrow \mathbb{Z}\langle E, F \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a, b, c \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle p, q \rangle \rightarrow 0$$

$$E \mapsto a + c - b$$

$$F \mapsto b + c - a$$

$$a \mapsto q - p$$

$$b \mapsto q - p$$

$$c \mapsto q - q = 0$$

$$H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\mathbb{Z}\langle p, q \rangle}{\langle q - p \rangle} \cong \mathbb{Z}$$

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\mathbb{Z}\langle a - b, c \rangle}{\langle (a - b) + c, (b - a) + c \rangle} \cong \frac{\mathbb{Z}\langle a - b, c \rangle}{\langle (a - b) + c, 2c \rangle}$$

$$(a - b) + c + (b - a) + c = 2c$$

$$\cong \frac{\mathbb{Z}\langle (a - b) + c, c \rangle}{\langle (a - b) + c, 2c \rangle} \cong \mathbb{Z}_2\mathbb{Z} \quad (\text{Wow!})$$

$$H_2^\Delta(X) = \frac{\ker \partial_2}{\operatorname{im} \partial_1} = \frac{0}{0} = 0 \quad \cup$$

$$\ker \partial_2 = ? \quad \partial E = (a - b) + c \quad \partial F = (b - a) + c$$

No linear combo of E and F is in the kernel!

$$H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}_2\mathbb{Z} & n=1 \\ 0 & \text{o/w} \end{cases}$$

Rmk. It may be disturbing that $H_2^\Delta(X) = 0$, since X is a 2D

Object! It turns out that the solution is to use different

coefficients: $(a - b) + c = (b - a) + c$ over $\mathbb{F}_2$!

Quick Start: classification of surfaces

No proof; just know the statement and be able to draw the cell decomposition (ie. polygon depiction of quotient space)

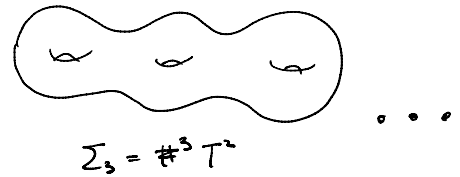
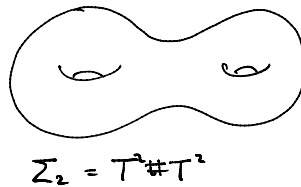
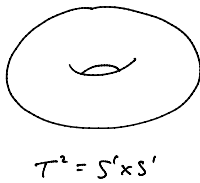
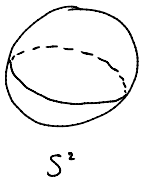
An unfortunately overloaded term:

defn A manifold is closed if it is compact and has empty boundary. (This definition is intrinsic to the space — there is no embedding in an ambient $\mathbb{R}^N$!)

thm. Every closed, connected, orientable surface is homeomorphic to one of the spaces in this list:

g = genus

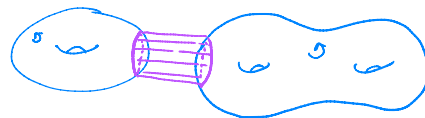
Euler char = $\chi = 2g - 2$.



Connected sum (#) of two oriented manifolds



Delete nbhds of points



Identify boundaries

(orientation reversing on the ∂ so that orientations on the two halves are preserved)

Result:



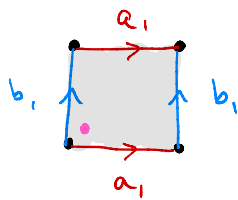
Construction

For $g \geq 1$, the surface (i.e. 2-manifold) Σ_g is a quotient space of the $4g$ -gon with identifications given by the word

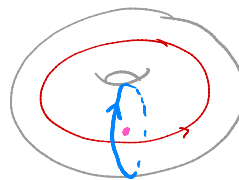
$$\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = \prod_{i=1}^g \underbrace{[a_i, b_i]}_{\substack{\uparrow \\ \text{The commutator of } a_i \text{ and } b_i}}$$

First two examples (pg 5 Hatcher)

$g=1$

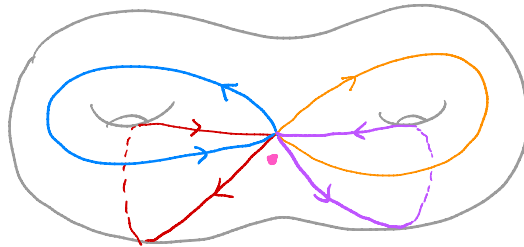
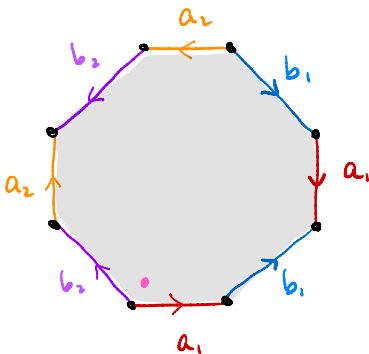


glue
~



Boundary word: $a_1 b_1 a_1^{-1} b_1^{-1}$

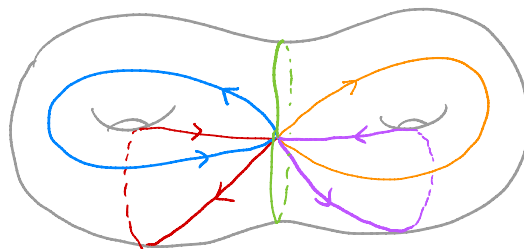
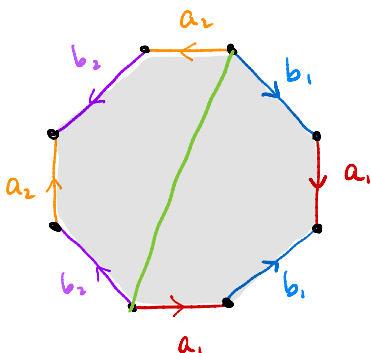
$g=2$



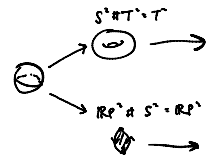
Boundary word: $(a_1 b_1 a_1^{-1} b_1^{-1}) (a_2 b_2 a_2^{-1} b_2^{-1})$

Why does this work? Idea:

green = ∂ of connected sum region!



We have a corresponding theorem & construction



thm Every closed, connected, non-orientable surface is homeo to one of the spaces in this list:

γ = non-orientable genus

$$\chi = 2 - \gamma$$

(sphere w/ γ "cross-caps")

$$\gamma = 1$$

$$\mathbb{RP}^2$$

$$\gamma = 2$$

$$\mathbb{RP}^2 \# \mathbb{RP}^2$$

$$\gamma = 3$$

$$\#^3 \mathbb{RP}^2$$

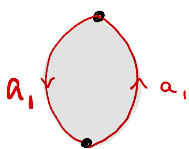
... and so on

(Klein bottle!)

Construction

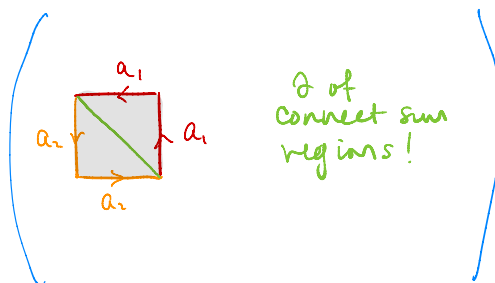
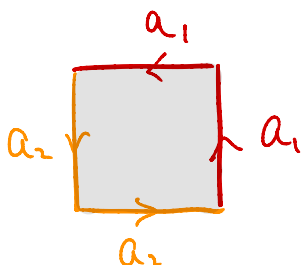
For $\gamma \geq 1$, the non-orientable surface $\#^\gamma \mathbb{RP}^2$ is a quotient space of the 2γ -gon with identifications given by the word $\prod_{i=1}^{\gamma} a_i^2$.

$\mathbb{RP}^2$:



$\mathbb{RP}^2 \# \mathbb{RP}^2$:

Klein bottle!



CW-complex (aka. cell complex, cellular complex)

pg. 5 of Hatcher

defn The disk / ball of dim n , denoted D^n or B^n , is
 $\{p \in \mathbb{R}^n \mid |p| \leq 1\}$ "n-ball"

The sphere of dim n , denoted S^n , is

$$\{p \in \mathbb{R}^{n+1} \mid |p| = 1\}$$
 "n-sphere"

Instead of simplices, we build spaces from cells:

0-cell: $e^0 = \bullet$

a point.

$$\partial e^0 = \emptyset = S^{-1}$$

1-cell: $e^1 = \text{---}$

an interval.

$$\partial e^1 = \bullet \quad \bullet = S^0 \text{ (two points)}$$

2-cell: $e^2 = \text{---}$

a disk (D^2)

$$\partial e^2 = \bigcirc = S^1$$

n-cell $e^n \cong D^n$

$$\partial e^n = S^{n-1}$$

For $n \geq 1$, ∂e^n is glued onto a complex of smaller dimension. (less strict gluing rules!)

Construction of cell complex:

(1) Start with a discrete set of 0-cells X^0 .

(X^0 is the 0-skeleton)

(2) Inductively build the n -skeleton X^n from the $(n-1)$ -skeleton X^{n-1} by attaching n -cells e_α^n to X^{n-1} , via maps $\varphi_\alpha: \partial e^n \cong S^{n-1} \longrightarrow X^{n-1}$

(no extra restrictions*!!!)

* More details in Hatcher.

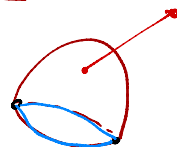
OK ✓



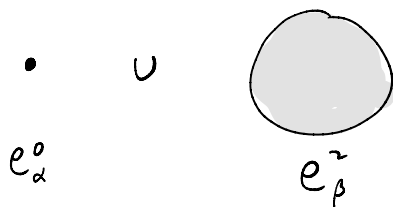
BAD



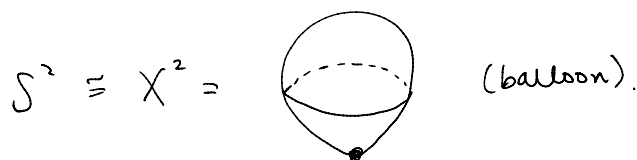
BAD



eg. we can build S^2 with just two cells:



$\varphi_\beta: \partial e_\beta^2 \longrightarrow X^0$ crushes S^1 to a pt.



Rmk. There is CW homology, and honestly its how we usually compute homology of spaces.

Will talk about later, when we have more generality...

We focus so hard on simplicial homology b/c its relationship to singular homology, which is one of the most popular ways homology shows up in proofs...

↳ I'll hopefully give an example to support this later in the course...

Simplicial Homology

defn Given a topological space X : (very non-restrictive!)

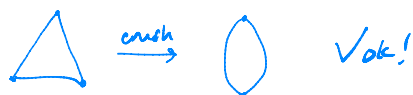
singular chain cpx: $\dots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$.

- (singular) n -chains:

$$C_n = \mathbb{Z} \langle \{ \text{continuous maps } \sigma_\alpha: \Delta^n \rightarrow X \} \rangle$$

— that's a lot (uncountable, usually)

— no homeo (or interior) condition! $\Rightarrow$ "singular"



- differential ("boundary map") same! on generators:

$$\partial \sigma_\alpha = \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Indeed, restriction is a $n-1$ chain!

prop. $\partial^2 = 0$

pf. Same as in simplicial case!

$\hookrightarrow$ Mind blowing...

$S(X)$ is a Δ -cpx...

defn. Singular homology of X : $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Example? we can do $X = *$ (a point!) $\rightarrow$ later today; motivates reduced homology.

Proofs instead of examples!

Flatcher discussion: geometric interpretation...

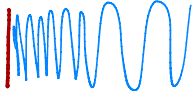
Observation:

prop. 2.6 If $X = \coprod_{\alpha \in A} X_\alpha$ where the X_α are the *path* components,
then $H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha)$ for all n .

Pf.

Key: im ∂_2 always path contd!

The $C_n(X)$, ∂_n both split as direct sum. (Hence as ker, im, and homology) //

Q. $H_*(X)$ where $X =$  ?

y axis $\cup$ top sine?
b/w $[-1, 1]$