

Recall Defn of singular homology

- $C_n = \mathbb{Z} \langle \{ \text{continuous maps } \sigma_\alpha: \Delta^n \rightarrow X \} \rangle$
- differential ("boundary map")

$$\partial \sigma_\alpha = \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

prop 2.7 If $X \neq \emptyset$ and is path-contd, then $H_0(X) \cong \mathbb{Z}$.

↳ Cor If $X = \coprod_{\alpha \in A} X_\alpha$ (path cpts) then $H_0(X) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$

Pf.

$$H_0(X) = C_0(X) / \text{im } \partial_1$$

Def. (an augmentation!) $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$

$$\sum_i c_i \sigma_i \mapsto \sum_i c_i$$

— clearly a surjective hom.

Claim If X is path-contd, $\ker \varepsilon = \text{im } \partial_1$.

↳ $\Rightarrow \varepsilon: C_0(X) \rightarrow \mathbb{Z}$ induces an isom

$$\varepsilon_*: C_0(X) / \ker \varepsilon \rightarrow \mathbb{Z}$$

$$= C_0(X) / \text{im } \partial_1$$

$$= H_0(X)$$

(Some isom then...)

$$f: M \rightarrow N$$

$$f_*: M / \ker f \xrightarrow{\cong} N$$

ε_* is isom $H_0(X) \cong \mathbb{Z}$.

pf of claim WTS $\ker \varepsilon = \text{im } \partial$. Double inclusion!

$$\boxed{\text{Im } \partial \subset \ker \varepsilon} \quad \text{given } \sigma: \Delta' \rightarrow X,$$

$$\varepsilon \partial_1(\sigma) = \varepsilon(\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_1]}) = 1 - 1 = 0$$

$$\boxed{\ker \varepsilon \subset \text{im } \partial.}$$

Some genuine 0-chain

$$\text{Suppose } \varepsilon\left(\sum_i c_i \sigma_i\right) = 0, \text{ i.e. } \sum c_i = 0.$$

Each σ_i is a sing. 0-simplex $\sigma_i: \Delta^0 \text{ pt} \rightarrow X$.

Setup. Choose basepoint $x_0 \in X$.

Let σ_0 be the singular 0-simplex
w/ image the basepoint x_0 .

red.
basepoints:
will let us define
reduced hom.
"based top spaces"

Since X is path ctd, we can choose a path Δ'

$$\tau_i: [0, 1] \rightarrow X$$

$$\text{as 1-simplex: } \tau_i: [v_0, v_1] \rightarrow X$$

from the chosen basepoint x_0 to $\sigma_i(v_0)$.

↳ Observe that τ_i is a sing. 1-simplex!

$$\text{Then } \partial \tau_i = \sigma_i - \sigma_0.$$

$$\sum c_i = 0 \Rightarrow \text{this is 0!}$$

$$\text{so... } \partial\left(\sum_i c_i \tau_i\right) = \sum_i c_i \sigma_i - \underbrace{\sum_i c_i \sigma_0}_{\text{a totally legit 1-chain!}} = \sum_i c_i \sigma_i \quad (!!!)$$

$$\Rightarrow \sum_i c_i \sigma_i \text{ is a boundary (recall: in im } \partial).$$

Reduced homology

defn let X be a nonempty space.

The reduced homology of X , $\tilde{H}_*(X)$, is
is the homology of the augmented chain cpx

$$\longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} \underline{C_0(X)} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

(where $\tilde{H}_0(X) := \ker(\varepsilon) / \operatorname{im} \partial_1$.)

Remk. ① X can have multiple path cpts.

* $\oplus H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$, and $H_n(X) = \tilde{H}_n(X) \quad \forall n > 0$.

rmk suspensions + homology grs.

eg. S spectrum

Chain maps

(abs. defs later)

alg

defn Given $\mathcal{C} = (\mathcal{C}, \partial)$ and $\mathcal{C}' = (\mathcal{C}', \partial')$, a chain map

$g: \mathcal{C} \rightarrow \mathcal{C}'$ is a collection of maps $\{g_i: C_i \rightarrow C'_i\}$

that commutes with the diff's:

$$\begin{array}{ccc} C_i & \xrightarrow{\partial_i} & C_{i-1} \\ g_i \downarrow & \curvearrowright & \downarrow g_{i+1} \\ C'_i & \xrightarrow{\partial'_i} & C'_{i-1} \end{array}$$

"commutative diagram"

$$\partial' g = g \partial$$

top

! Try my best to use this notation!

Given spaces $X, Y \rightsquigarrow$ chain cpxs $C_*(X), C_*(Y)$.

A map $f: X \rightarrow Y$ of top spaces (???) (ie. cont maps!)

induces map on the chain cpxs:

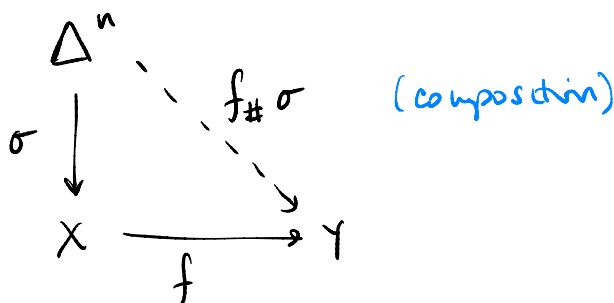
"f-sharp"

$$f_{\#} \partial(\sigma) = f_{\#} \left(\sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$$

$$= \sum_i (-1)^i f \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \partial f_{\#}(\sigma)$$

idea "induced"

the "obvious" related map.



Pictorially at chain level:

Keep on board

$$\begin{array}{ccccccc}
 \dots & \rightarrow & C_{i+1}(X) & \xrightarrow{\partial_{i+1}^X} & C_i(X) & \xrightarrow{\partial_i^X} & C_{i-1}(X) \rightarrow \dots \\
 & & \downarrow (f_{\#})_{i+1} & \curvearrowright & \downarrow (f_{\#})_i & \curvearrowright & \downarrow (f_{\#})_{i-1} \\
 \dots & \rightarrow & C_{i+1}(Y) & \xrightarrow{\partial_{i+1}^Y} & C_i(Y) & \xrightarrow{\partial_i^Y} & C_{i-1}(Y) \rightarrow \dots
 \end{array}$$

prop. chain maps induce maps on homology: core idea *****

pt.

- g takes cycles to cycles:

$$\underbrace{\partial g(\sigma)}_{\text{cycle in } C'} = g \underbrace{\partial(\sigma)}_{\text{cycle in } C} = g(0) = 0$$

- g takes boundaries to boundaries

$$g(\underbrace{\partial \sigma}_{\partial \text{ in } C}) = \underbrace{\partial(g\sigma)}_{\partial \text{ in } C'}$$

- $\Rightarrow$ can define $g_*: H_*(C) \rightarrow H_*(C')$ by

$$g_*([\sigma]) = [g\sigma] \quad \leftarrow \text{equiv class}$$

$\hookrightarrow$ well-defn b/c if $\sigma, \tau \in [\sigma]$, then

$$\tau - \sigma = \partial \beta \quad (\text{for some } \beta \in C)$$

$$\text{and so } g(\tau - \sigma) = g\partial \beta$$

$$\Rightarrow [g\tau] - [g\sigma] = [0] \Rightarrow [g\tau] = [g\sigma]$$

□

notation $f: X \rightarrow Y$ mor in Top.

$f_{\#}: C_*(X) \rightarrow C_*(Y)$ mor in $Ch Grp$

Write

$f_*: H_*(X) \rightarrow H_*(Y)$ mor in $g\text{-}Z\text{mod}$

for the induced map in homology.

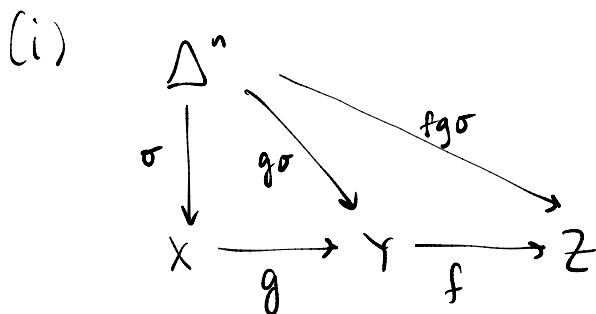
More observations: (also algebraic result)

prop: (i) $(fg)_* = f_* g_*$ for $X \xrightarrow{g} Y \xrightarrow{f} Z$

"covariant functor"

(ii) $1_* = 1$ where 1 is identity map

pf sketch.



(ii) clear!

//

chain homotopy (Write topo. item first - next page)

$$\begin{array}{ccccccc}
 \dots & \rightarrow & C_{i+1}(X) & \xrightarrow{\partial^X} & C_i(X) & \xrightarrow{\partial_i^X} & C_{i-1}(X) \rightarrow \dots \\
 & & \searrow f_{\#} \downarrow g_{\#} & & \searrow f_{\#} \downarrow g_{\#} & & \searrow f_{\#} \downarrow g_{\#} \\
 \dots & \rightarrow & C_{i+1}(Y) & \xrightarrow{\partial_{i+1}^Y} & C_i(Y) & \xrightarrow{\partial_i^Y} & C_{i-1}(Y) \rightarrow \dots
 \end{array}$$

Red arrows labeled h_i and h_{i-1} point from $C_{i+1}(X)$ to $C_i(Y)$ and from $C_i(X)$ to $C_{i-1}(Y)$ respectively.

defn A chain htpy b/w chain maps

$f, g: C \rightarrow C'$ is a collection of maps

$h = \{ h_i: C_i \rightarrow C'_{i+1} \}$ such that

$$f_i - g_i = \partial h + h \partial.$$

chain htpy
condition

prop If f, g are chain htpic (ie $\exists$ htpy h)

then $f_* = g_*$ (ie same homomorphism on H_*).

→ save for later. (Maybe HW).

Important, less trivially proven:

thm 2.10 Homotopic maps $f, g : X \rightarrow Y$ induce the same hom $f_* = g_* : H_*(X) \rightarrow H_*(Y)$.

Cor 2.11 If $f : X \rightarrow Y$ is a htpy equivalence

then $f_* : H_*(X) \rightarrow H_*(Y)$ is an isom of graded $\mathbb{Z}$ -mods.

ie. $f_n : H_n(X) \rightarrow H_n(Y)$ is an isom $\forall n$.

Cor. If X is contractible, then $H_*(X) = 0$

where $0 \in \text{graded } \mathbb{Z}\text{-mod.}$

Pf. (Sketch - get the main idea + tools in proof)
(Discussion 3)

defn chain htpy equiv:

$$\exists g : Y \rightarrow X \text{ s.t. } fg \simeq \text{id}_Y, gf \simeq \text{id}_X$$

where $\simeq$ means "chain homotopic to".

Discussion 3

① proof of htpy invariance

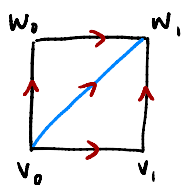
Recall Algebraically we want to see a chain htpy $h = \{h_i\}$ as below:

$$\begin{array}{ccccc}
 C_{i+1}(X) & \xrightarrow{\quad} & C_i(X) & \xrightarrow{\quad} & C_{i-1}(X) \\
 \downarrow (f\#)_i & \searrow (g\#)_{i+1} & \downarrow (f\#)_i & \searrow (g\#)_i & \downarrow (f\#)_{i-1} \\
 C_{i+1}(Y) & \xrightarrow{\quad} & C_i(Y) & \xrightarrow{\quad} & C_{i-1}(Y)
 \end{array}$$

The diagram illustrates the chain homotopy $h = \{h_i\}$ between the chain complexes $C_{\bullet}(X)$ and $C_{\bullet}(Y)$. The horizontal arrows represent the differentials, and the vertical arrows represent the maps $f\#$ and $g\#$. The diagonal arrows represent the chain homotopy maps h_i .

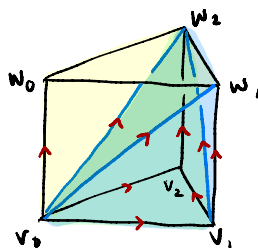
How to: Divide $\Delta^n \times I$ into $(n+1)$ -simplices:

eg. $n=1$



$n=2$

"prism!"



* The ordering is lexicographic with
 $v < w, v_i < v_{i+1}, w_i < w_{i+1}.$

To do this carefully, need to talk about barycentric coordinates.

see Hatcher for argument w/ coordinates, more rigorous proof.

- In general, $\Delta^n \times I$ has "bottom" $\Delta^n \times \{0\} = [v_0, \dots, v_n]$
"top" $\Delta^n \times \{1\} = [w_0, \dots, w_n].$

- We have all the vertices + need to define collection of Δ^{n+1} simplices. given by the ordered sets

$$[v_0, \dots, v_i, w_i, \dots, w_n]$$

if you keep the ordering then

- the chosen $(n+1)$ simplices won't overlap.
- the result is a simplicial complex.

! Need to check that every point is in a simplex

pf of thm 2.10

$f, g: X \rightarrow Y$ htpc.

Given a htpy $F: X \times I \rightarrow Y$ from f to g , define (chain homotopies)

prism operators $P: C_n(X) \rightarrow C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}_I) \Big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

$\downarrow$
 $\sigma: \Delta^n \rightarrow X$

$\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$

claim $\partial P = g_{\#} - f_{\#} - P\partial$ ($g_{\#} - f_{\#} = \partial P + P\partial$)

$\swarrow$
geom: the ∂ of
the big prism

$\downarrow$
top
 $\Delta^n \times \{1\}$

$\downarrow$
bot
 $\Delta^n \times \{0\}$

$\searrow$
sides $\partial \Delta^n \times I$

pf of claim calculation:

$\partial P(\sigma) =$

$$\sum_i \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1}_I) \Big|_{[v_0, \dots, \hat{v}_i, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_i \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathbb{1}_I) \Big|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

• The case $i=j$ (quick check)

$$F \circ (\sigma \times 1) |_{[\hat{v}_0, w_0, \dots, w_n]} = g \circ \sigma = g \# \sigma$$

$$F \circ (\sigma \times 1) |_{[v_0, v_1, \dots, v_n, \hat{w}_n]} = -f \circ \sigma = -f \# \sigma$$

• otherwise they cancel (w/ opp signs)

$$\text{eg. } [v_0, \hat{v}_1, w_1, w_2, \dots, w_n] = [v_0, \hat{w}_0, w_1, \dots, w_n]$$

• The remaining terms ($i \neq j$) are exactly $-P\partial(\sigma)$:

$$\sigma \in \mathcal{C}_n(X)$$

$$\partial_n \sigma = \sum_j (-1)^j \sigma |_{[v_0, \dots, \hat{v}_j, \dots, v_n]}$$

$$P_{n-1} \circ \partial_n \sigma = \sum_j (-1)^j \underbrace{P_{n-1}(\sigma |_{[v_0, \dots, \hat{v}_j, \dots, v_n]})}_{\text{blue underline}}$$

$$= \sum_{i < j} (-1)^i (-1)^j \underbrace{F \circ (\sigma |_{[v_0, \dots, \hat{v}_j, \dots, v_n]} \times 1) |_{[v_0, \dots, v_i, w_i, \dots, w_n]}}_{\text{red underline}}$$

$$= (\sigma \times 1) |_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_{i < j} (-1)^i (-1)^{j-1} \underbrace{F \circ (\sigma |_{[v_0, \dots, \hat{v}_j, \dots, v_n]} \times 1) |_{[v_0, \dots, v_i, w_i, \dots, w_n]}}_{\text{red underline}}$$

$$= (\sigma \times 1) |_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

* w_j is geometrically at " $v_j \times 1$ "

Compare and verify that indeed $-P\partial$ accounts for the remaining terms in ∂P .

* geometric interpretation aside

To conclude the proof, prove that if $\exists$ chain htpy h b/w chain maps $f_\#$ and $g_\#$, then $f_\# = g_\#$. (pure algebra)



We like sub, quotient objects b/c then we can build more objects from existing ones (eg. modules, spaces...)
 → define these explicitly? (Done in class)

Goal Relate $H_*(A)$, $H_*(X)$, and $H_*(X/A)$:

Thm 2.13

If X is a space and
 A is a nonempty closed subspace
 that is a deformation retract
 of some nbhd in X , } good pair (X, A)

then there is an exact sequence

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{j_*} & \tilde{H}_n(X/A) \\
 & & & & & & \searrow \delta \\
 & & \tilde{H}_{n-1}(A) & \xrightarrow{i_*} & \tilde{H}_{n-1}(X) & \rightarrow & \cdots \\
 & & & & & & \searrow \\
 & & & & \cdots & \rightarrow & \tilde{H}_0(X/A) = 0
 \end{array}$$

LES induced by SES.

Exact sequences

works in great generality,
eg. Abelian cats.

defn A sequence of morphisms

$$\dots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \dots$$

Has
examples

is exact if $\forall n, \ker \alpha_n = \operatorname{im} \alpha_{n+1}$.

Note Viewed as chain cpx in particular, this (A, ∂)

where $A = \bigoplus A_k$, $\partial = \sum f_k$ is acyclic, ie has
trivial homology.

Useful idea for characterizing maps. Eg:

- $0 \rightarrow A \xrightarrow{\alpha} B$ is exact iff α is injective
- $A \xrightarrow{\alpha} B \rightarrow 0$ iff α is surj.

defn Short exact sequence (SES):

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

note α is inj, β is surjective

} suggestively:
 A, B, C are
going to be chain
cpxs; α, β chain
maps

Long Exact sequences (LES) from SESs of chain complexes

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{A} & \cdots \longrightarrow & A_{n+1} & \xrightarrow{\partial^A} & A_n & \xrightarrow{\partial^A} & A_{n-1} \longrightarrow \cdots \\
 i \downarrow & & i \downarrow & & i \downarrow & & i \downarrow \\
 \mathcal{B} & \cdots \longrightarrow & B_{n+1} & \xrightarrow{\partial^B} & B_n & \xrightarrow{\partial^B} & B_{n-1} \longrightarrow \cdots \\
 p \downarrow & & p \downarrow & & p \downarrow & & p \downarrow \\
 \mathcal{C} & \cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial^C} & C_n & \xrightarrow{\partial^C} & C_{n-1} \longrightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Keep on Board

SES of chain complexes $\rightarrow$ $\mathbb{Z}$ -many SES,
 plus the maps i, p commute w/ the
 differentials $\partial^A, \partial^B, \partial^C$.

- These i, p induce maps on homology, called i_*, p_* .

Picture

$$\begin{array}{ccc}
 H_*(\mathcal{A}) & \xrightarrow{i_*} & H_*(\mathcal{B}) \\
 \swarrow \text{"}\partial\text{"} & \nearrow p_* & \\
 & H_*(\mathcal{C}) &
 \end{array}$$

- We can also define (and define it well!) a

connecting homomorphism

$$\partial: H_n(\mathcal{C}) \longrightarrow H_{n-1}(A) \quad \text{homgr preserving.}$$

$$\partial = \{ \partial_n: H_n(\mathcal{C}) \longrightarrow H_{n-1}(A) \}$$

* very unfortunate clashing of notation with diff'l of $A, B, \mathcal{C}$; here carelessly wrote $\partial^A, \partial^B, \partial^C$

* This is indeed standard notation, so I will not change it.

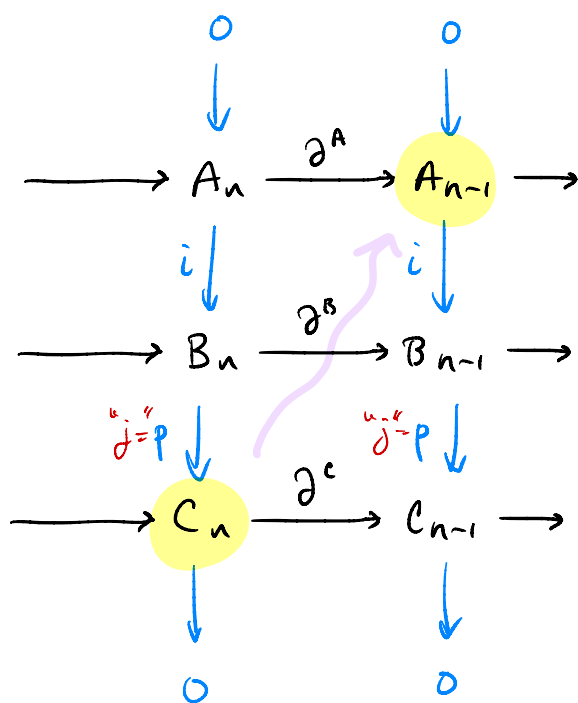
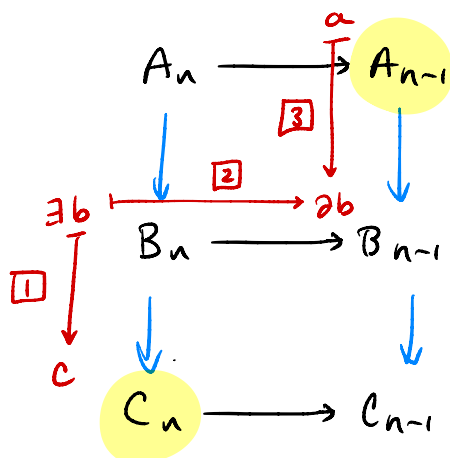


Diagram Chasing!

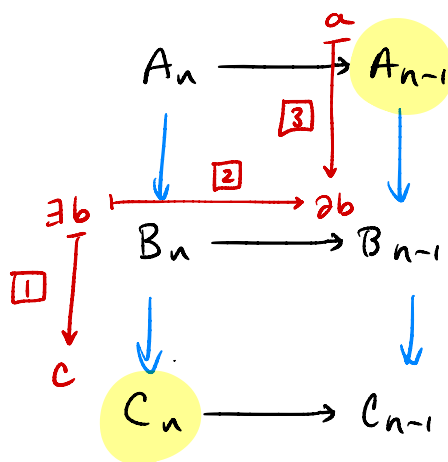
We wish to define $\partial[c] = [a]$ where a is chosen as below:



Things to prove:

- I. Given $c \in C_n$, find some a
- II. show different choices in the defn of a lead to a' in the same homology class
- III. show that different choices of $c \in [c]$ yield the same $[a]$.
- IV. ∂ is a homomorphism (easy)

I. let $c \in [c]$. * note c is a cycle



[1] p is surjective

[2] use ∂ to map forward

[3] c is a cycle $\Rightarrow \partial^2 c = 0$

• Bottom square commutes $\Rightarrow p(\partial b) = 0$.

u. $\partial b \in \ker p$.

• Right column is exact $\Rightarrow \partial b \in \text{im } i$

$\Rightarrow \exists a$ s.t. $i(a) = \partial b$.

II. the choices made in I do not matter:

[1] i is injective so there was no choice made

[2] made no choices

[3] Suppose we chose a different lift $b' \in p^{-1}(c)$.

Then $b - b' = i(a')$ for some $a' \in A_n$ (since $\ker p = \text{im } i$).

Then $\partial b' = \partial(b - i(a')) = \partial b + \partial i(a')$.

Since i is a chain map, $\partial i(a') = i \partial a'$

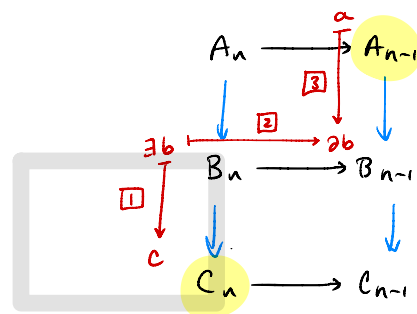
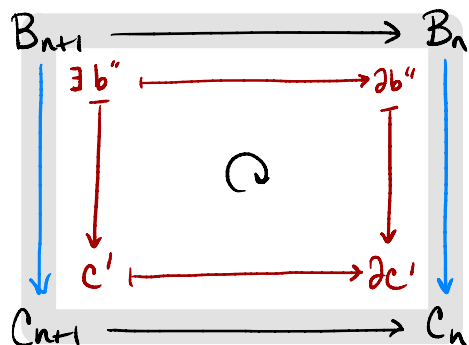
$\Rightarrow$ we obtain $a + \partial a' \in [a]$.

III. Our choice of $c \in [c]$ did not matter either.

Any other choice of representative would be of the form

$$c + \partial c' \text{ where } c' \in C_{n+1}.$$

I



May choose $b + \partial b'' \in B_n$.

$$\text{Then } \partial(b + \partial b'') = \partial b + 0.$$

Done! Choice $c + \partial c' \in [c]$ did not matter!

IV. ∂ is a homomorphism (easy)

left to reader ☺

main thm for today:

thm 2.16 The sequence of homology groups

$$\begin{array}{ccccccc} \cdots & \rightarrow & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) \\ & & & & & & \searrow \\ & & & & & & \hookrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C) \\ & & & & & & \searrow \\ & & & & & & \hookrightarrow \cdots \end{array}$$

is exact.

pf.

Q. What do we need to check?

im $\subset$ ker statements:

① $j_* i_* = 0$

② $\partial j_* = 0$

③ $i_* \partial = 0$

ker $\subset$ im statements

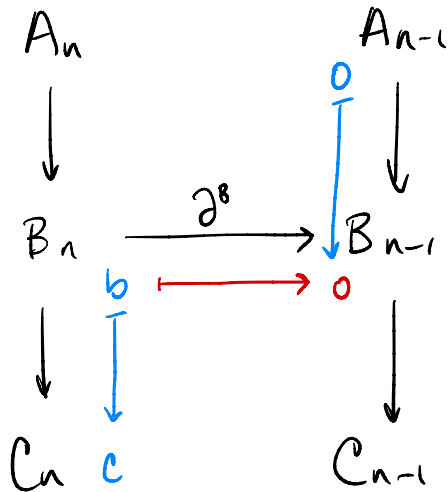
④ $\ker j_* \subset \operatorname{im} i_*$

⑤ $\ker \partial \subset \operatorname{im} j_*$

⑥ $\ker i_* \subset \operatorname{im} \partial$

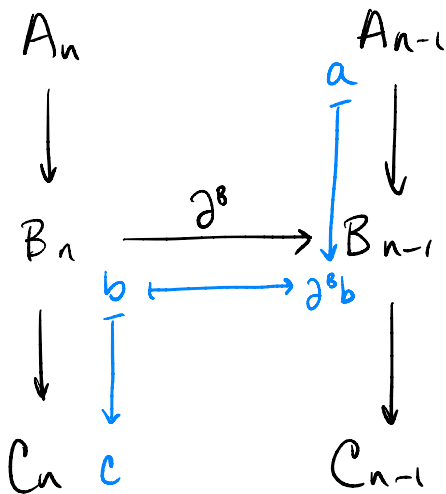
① $j \circ i = 0 \implies j_* \circ i_* = 0$ homology is a (covariant functor)

② $\partial j_* = 0$ If $[b] \in H_n(B)$ then b is a cycle $\implies \partial^B(b) = 0$



$j_*[b] \implies \partial[c] = 0.$

③ $i_* \partial = 0$



$[c] \xrightarrow{\partial} [a] \xrightarrow{i_*} [\partial^B b] = 0$

④ $\ker j_* \subset \operatorname{im} i_*$

let $[b] \in \ker j_*$

$\Rightarrow [jb] = 0$

$\Rightarrow jb = \partial c'$ for some $c' \in C_{n+1}$

j is surjective $\Rightarrow c' = j(b')$ f.s. $b' \in B_{n+1}$.

Now... $j(b - \partial b') = j(b) - j(\partial b') \stackrel{\text{ch map}}{=} j(b) - \partial j(b') = 0$

$\text{b/c } \partial j(b') = \partial c' = j(b).$

$\Rightarrow b - \partial b' \in \ker j = \operatorname{im} i \Rightarrow b - \partial b' = i(a)$ f.s. $a \in A_n$.

a is a cycle:

$i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$

and i is injective.

So $i_*[a] = [b - \partial b'] = [b] - [\partial b'] = [b].$

$\Rightarrow$ we have found $[a] \in H_n(A)$ that maps onto arbitrary $[b] \in \ker j_*$.

⑤ $\ker \partial \subset \operatorname{im} j_*$

⑥ $\ker i_* \subset \operatorname{im} \partial$

Similar flavor, not instructive for me to walk through in class

* highly recommended that you work through these on your own for your own notes/practice.

(HW)

