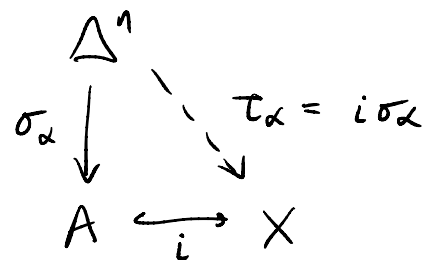


Relative Homology Groups

Obs. We can view $C_n(A) \hookrightarrow C_n(X)$
 $\sigma_\alpha \mapsto \tau_\alpha$



Injective: If $i\sigma_\alpha = i\sigma_\beta$, then $\sigma_\alpha = \sigma_\beta$.

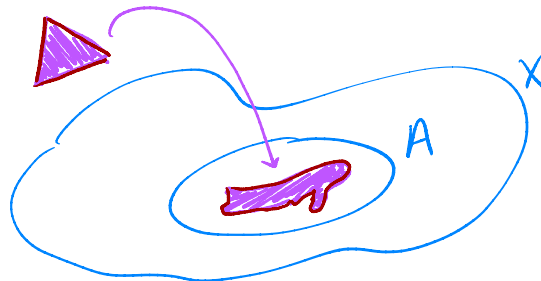
defn thm (Relative Homology)

- Given $A \subset X$, let $C_n(X, A) := C(X) / C(A)$
- The bd. map $\partial_x: C_n(X) \rightarrow C_{n-1}(X)$ induces

$$\partial: C_n(X, A) \rightarrow C_{n-1}(X, A)$$

since $\partial_x(C_n(A)) \subseteq C_{n-1}(A)$

think about it...



So we may define the relative singular chain cx :

$$\cdots \rightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$

• $\partial^2 = 0$ since $\partial_x^2 = 0$ already.

• relative cycles:

$$n\text{-chains } \tau \in C_n(X) \text{ s.t. } \partial \tau \in C_{n-1}(A)$$

• relative boundaries:

$$\tau = \partial \tau_1 + \tau_2 \text{ where } \begin{matrix} \tau_1 \in C_{n+1}(X) \\ \tau_2 \in C_n(A) \end{matrix}$$

Thm 2.13 If (X, A) is a good pair,

X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood of X .

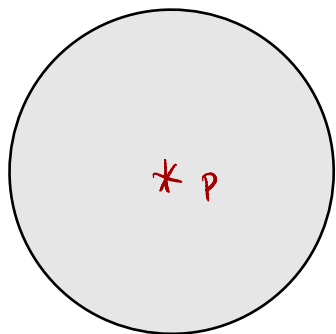
then there is an exact sequence

$$\begin{array}{ccccccc} \cdots & \rightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{j_*} & \tilde{H}_n(X/A) \xrightarrow{\partial} \\ & & & & & & \uparrow \text{LES induced by SES.} \\ & & \tilde{H}_{n-1}(A) & \xrightarrow{i_*} & \tilde{H}_{n-1}(X) & \rightarrow & \cdots \\ & & & & & & \uparrow \\ & & & & \cdots & \rightarrow & \tilde{H}_0(X/A) \rightarrow 0 \end{array}$$

(see remark later.)

Rule Where do we use good pair? We'll see Excision Thm next, whose pt is more geometric. Then we'll see that when (X, A) is good, $H_*(X/A) = H_*(X, A)$.

eg. Not good pair: ($A \subset X$ open)

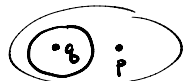


$$A = D - \{p\} =: A$$

"annulus"

$$X = D$$

$$X/A \cong \{q, p\} \text{ with topology}$$



"Sierpiński space"

(non Hausdorff!)

If we applied the LES we would get

$$\tilde{H}_2(A) \longrightarrow \tilde{H}_2(D) \longrightarrow \tilde{H}_2(D, A) \Rightarrow \mathbb{Z} \dots$$

algebra: $H_n(D, A)$

topo: $H_n(D/A)$

discrepancy when (X, A) is not good.

$$H_*(X/A) \neq H_*(X, A)$$

in this case!

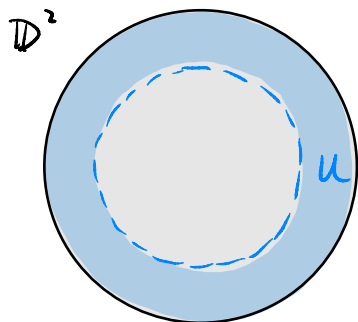
$$\tilde{H}_1(A) \longrightarrow \tilde{H}_1(D) \longrightarrow \tilde{H}_1(D, A) \Rightarrow 0$$

$$\tilde{H}_0(A) \longrightarrow \tilde{H}_0(D) \longrightarrow \tilde{H}_0(D, A) \longrightarrow 0$$

$\Rightarrow 0$ actually $\neq$ path $p \rightsquigarrow q$ ✓.

eg. good pair $S' \hookrightarrow D^2 \rightarrow S^2$

The calculation above is now valid.



$S' = \partial D^2$ is
def retract of
open nbhd $U \subset D^2$.

* some app to fixed pts skipped
Cor. 2.15.

LES odds + ends

① In the LES of a pair, $\partial[\alpha] = [\partial\alpha]$:

$$\begin{array}{ccc}
 C_n(A) & & \partial\alpha \\
 \downarrow & & \downarrow \\
 C_n(X) & \xrightarrow{\quad} & \partial\alpha \\
 \downarrow & & \\
 C_n(X, A) & & \alpha \\
 & & \downarrow \\
 & & \alpha \text{ rel. cycle} \\
 & & \Rightarrow \partial\alpha \in C_{n-1}(A)
 \end{array}$$

② There are two versions: un-reduced, and reduced
thm 2.16 (algebra)

following, we have

• LES for un-reduced hom of pair

$$\dots \rightarrow H_0(X, A) \rightarrow 0$$

• LES for reduced hom when $A \neq \emptyset$.

Have $\text{SES } 0 \rightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \rightarrow 0 \rightarrow 0$

$C_1(A) \quad C_1(X) \quad C_1(X, A)$

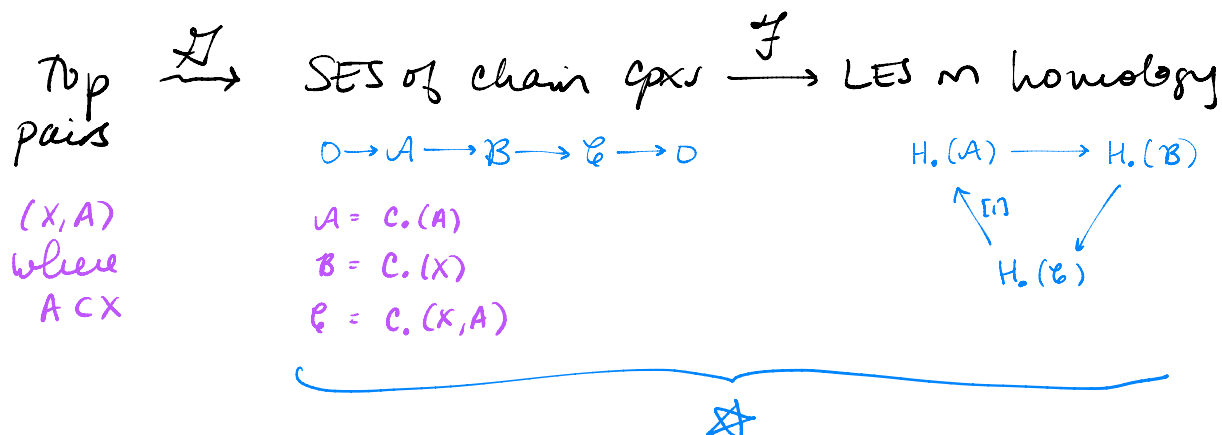
in dimension -1 .

$$\begin{array}{c}
 \rightarrow \tilde{H}_1(X, A) \\
 \hline
 \tilde{H}_0(A) \rightarrow \tilde{H}_0(X) \rightarrow \tilde{H}_0(X, A) \\
 \hline
 \rightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \rightarrow 0 \rightarrow 0
 \end{array}$$

$$\Rightarrow H_n(X, A) = \tilde{H}_n(X, A) \quad \forall n$$

Note on Naturality

We have a construction



defn/claim $\mathcal{F}$ is natural:

whenever we have a chain map

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \rightarrow 0 \\
 & & \downarrow \alpha & \curvearrowright & \downarrow \beta & \curvearrowright & \downarrow \gamma \\
 0 & \rightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C' \rightarrow 0
 \end{array}$$

(you could be more careful and write f_i^A, f_i^B, f_i^C etc.)

the induced maps in homology form a chain map:

$$\begin{array}{ccccccc}
 \dots & \rightarrow & H_n(A) & \rightarrow & H_n(B) & \rightarrow & H_n(C) \rightarrow \dots \\
 & & \downarrow \alpha_* & \curvearrowright & \downarrow \beta_* & \curvearrowright & \downarrow \gamma_* \\
 \dots & \rightarrow & H_n(A) & \rightarrow & H_n(B) & \rightarrow & H_n(C) \rightarrow \dots
 \end{array}$$

this map is actually $(\beta_*)_n$

This is again proven by diagram chasing (Hatcher pg 127, bottom).

* Rule in categories + functors.

We'll actually want to show that $f_* \circ \partial$ is natural:

Pairs of Top spaces, $\xrightarrow{f \circ \partial}$ LESs of $\mathbb{Z}$ -mods,
maps of pairs chain maps

We already showed maps of spaces $\leadsto$ chain maps (cat of $\mathbb{Z}$ -mod)

so squares w/ i_*, j_* commute.

ISTJ the square w/ ∂ commutes.

pf

we already have $f_{\#} \partial = \partial f_{\#}$ at the chain level
where ∂ is boundary map not
connecting map

let $[\alpha] \in H_n(X, A)$, rep'd by a relative cycle α .

Then

$$f_* \partial [\alpha] = f_* [\partial \alpha] = [f_{\#} \partial \alpha] = [\partial f_{\#} \alpha] = \partial f_* [\alpha]$$

$\uparrow$ connecting map
by pre. remark ①

//

We'll give a sketch. What you need to remember is the method of proof: mainly relies on LES and 5-lemma.

Some setup Let X be a Δ -complex,
let $A \subset X$ be a sub-complex.

① Relative simplicial homology
defined analogously: chain groups:

$$\Delta_n(X, A) = \Delta_n(X) / \Delta_n(A) \quad \text{relative simplicial chain}$$

② There is a canonical chain map

$$\varphi^{\text{rel}}: \Delta_*(X, A) \longrightarrow C_*(X, A)$$

inducing a canonical homomorphism

$$\varphi_*^{\text{rel}}: H_*^A(X, A) \longrightarrow H_*(X, A)$$

Let's write $\varphi^A: \Delta_*(A) \longrightarrow C_*(A)$ &

$$\varphi^X: \Delta_*(X) \longrightarrow C_*(X)$$

Homological algebra: 5-lemma (complete pt on HW03)

In any abelian category (have ker, coker, etc)
 (including $\text{Ch}(A)$, cat. of chain complexes)
 or abelian cat A .

we have the following algebraic lemma:

Lemma (The 5-lemma) If we have a comm. diag

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

where the rows are exact,

If $\alpha, \beta, \delta, \varepsilon$ are all isom., then γ is also an isom.

Pf. Simple diagram chasing (see HW03)

eg. Claim: β, δ surjective + ε injective
 $\Rightarrow \gamma$ is surjective.

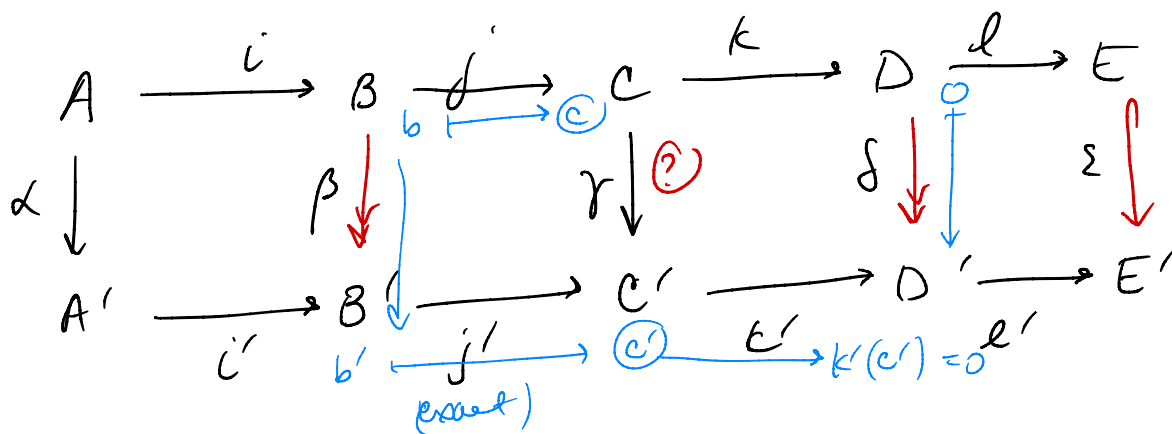
Pf.

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \textcircled{2} & & \delta \downarrow & & \varepsilon \downarrow \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

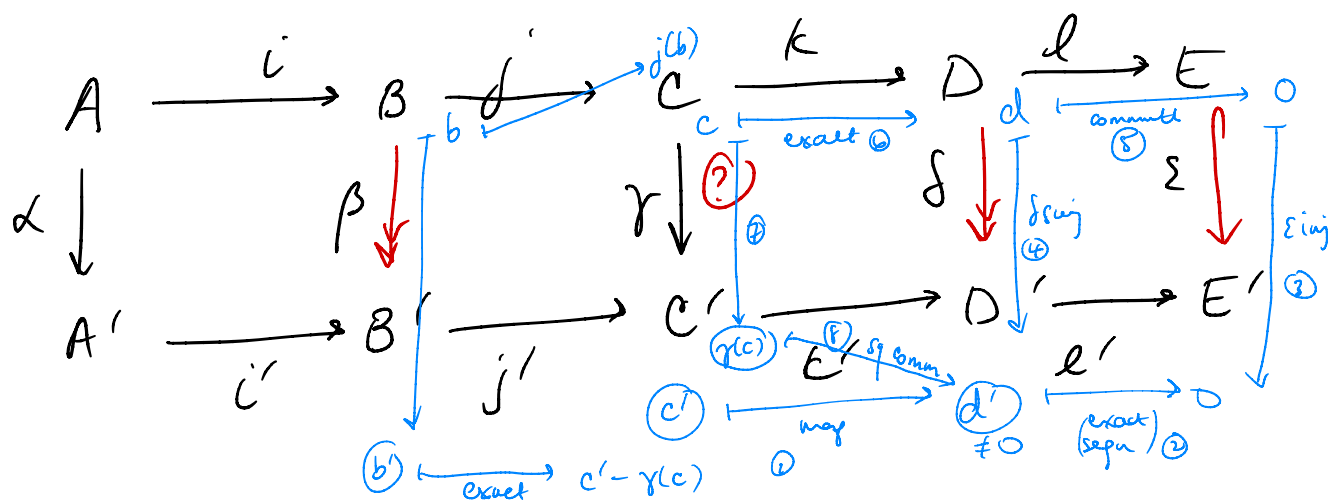
Let $c' \in C'$. WTF $c \in C$ s.t. $\gamma(c) = c'$.

wlog $c' \neq 0$.

Case 1 $k'(c') = 0$



Case 2 $k'(c') \neq 0$



Consider $c + j(b) \xrightarrow{\gamma} \gamma(c) + \gamma j(b)$
 $= j' \beta(b) = c' - \gamma(c)$
 $= c' \quad \checkmark$

HW: write out in words.

thm 2.27 let (X, A) be a Δ -pair.

Then $\varphi_n: H_n^\Delta(X, A) \rightarrow H_n(X, A)$ are isom
for all n

Pf sketch

case 1 X is finite dim, $A = \emptyset$.

let $X^k =$ the k -skeleton of X , consisting of all
simplices of dim k or less.

Consider the pair (X^k, X^{k-1}) . Get LES:

$n =$ homological degree

$k =$ skeleton dimension

Using naturality, we have a commutative diagram:

$$\begin{array}{ccccccccc}
 H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \rightarrow & H_n^\Delta(X^k) & \rightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \\
 \downarrow \varphi_{n+1}^{\text{rel}} & & \downarrow \varphi_n^X & & \downarrow \varphi_n^X & & \downarrow \varphi_n^{\text{rel}} & & \downarrow \varphi_{n-1}^X \\
 H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \rightarrow & H_n(X^k) & \rightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1})
 \end{array}$$

These maps are
easier to understand.

Want to use 5-lemma

Induct to get purple.

Step 1 show φ_* rel in the diag are $\cong$.

Key $\Delta_n(X^k, X^{k-1}) = 0$ when $n \neq k$

$$\Delta_k(X^k, X^{k-1}) = \mathbb{Z} \langle k\text{-simplices} \rangle$$

Same for $H_n^\Delta(X^k, X^{k-1})$.

Show same for $H_n(X^k, X^{k-1})$ (see next page). $\otimes$

(Induct on k)

Step 2 (Base) $H_n^\Delta(X^0) \rightarrow H_n(X^0)$ is isom. $\forall n$.

By induction may assume 2nd and 5th maps are isom

Step 3 Use 5-lemma.

//

Case 2 X is infinite dim

need to use fact that a cpct $C \subset X$ intersects only finitely many simplices

need to use top arguments to show φ_* is surj & inj.

Case 3 Relative

Use LFT. for (X, A)

(Already have $\cong$ for $H_n(X)$ and $H_n(A)$).

//

⊕ Claim $H_n(X^k, X^{k-1}) \cong \begin{cases} 0 & n \neq k \\ \mathbb{Z} \langle k\text{-simplices} \rangle & n = k \end{cases}$

We have characteristic maps for each k -simplex σ_α :

in X : $\phi_\alpha: \Delta_\alpha^k \rightarrow X$.

Combine these maps for k -simplices into a map

$$\Phi: \left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial \Delta_\alpha^k \right) \rightarrow (X^k, X^{k-1})$$

↑ topological ∂

Φ induces a homeomorphism of quotient spaces

$$\bar{\Phi}: \underbrace{\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial \Delta_\alpha^k}_{\cong \bigvee_\alpha S_\alpha^k} \xrightarrow{\cong} X^k / X^{k-1}$$

[Cor 2.25, to be covered] $\Rightarrow H_*(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha H_*(X_\alpha)$.

$$H_n(\bigvee_\alpha S_\alpha^k) \cong \begin{cases} \mathbb{Z} \langle k\text{-simplices} \rangle & n = k \\ 0 & n \neq k, 0 \\ \mathbb{Z} & n = 0 \text{ technically} \end{cases}$$

Hatcher's pt doesn't mention this.
But we can use the LES for reduced homology instead, and get $\tilde{H}_{0,*}^{\text{red}} = 0$ (an isom)
because X^k / X^{k-1} is path-cntd.

Fact. (See Example 2.23, 1st half)

$H_k(\Delta^k, \partial\Delta^{k-1})$ is generated by the identity map $\Delta^k \rightarrow \Delta^k$.

Now $\varphi_{k,k}^{\text{rel}} : \underbrace{H_k^{\Delta}(X^k, X^{k-1})}_{\substack{\text{gen by rel} \\ \text{cycles,} \\ \text{(inclusion maps)}}} \rightarrow \underbrace{H_k(X^k, X^{k-1})}_{\substack{\text{also gen by rel} \\ \text{cycles given} \\ \text{by the char} \\ \text{maps } \phi_{\alpha} : \Delta^k \rightarrow X^k}}$

So $\varphi_{k,k}^{\text{rel}}$ is an isom.

end pf of claim

More importantly:

From Case 1 we see that $H_n(X)$ is finitely generated
when $\Delta_n(X)$ is f.g.

F.g. module over $\mathbb{Z}$ (PID) are isom to $\mathbb{Z}^r \oplus \bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$

The rank (r) of $H_n(X)$ is called the n^{th} Betti number of X

The n_i are the torsion coefficients.