

⊗ Review claim from last time $H_n(X^k, X^{k-1})$

We still have htpy invariance even for $H_*(X, A)$:

prop 2.19 If two maps $f, g : (X, A) \longrightarrow (Y, B)$ are htpc
through maps of pairs $(X, A) \longrightarrow (Y, B)$,

then $f_* = g_* : H_*(X, A) \longrightarrow H_*(Y, B)$

Pf. Check that the prism operator induces
relative prism operator

$$P_n : C_n(X, A) \longrightarrow C_{n+1}(Y, B)$$

(check $C_n(A) \rightsquigarrow C_{n+1}(B)$ under P .)

Then in the quotient, the htpc condition

$$\partial P + P \partial = g_{\#} - f_{\#} \text{ still holds}$$

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Generalization: LES of a triple of spaces

defn Triple: (X, A, B) where $B \subset A \subset X$.
thm.

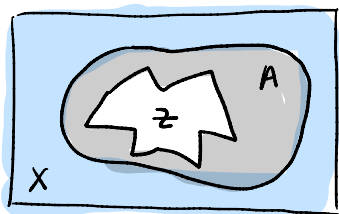
$$\begin{array}{ccc} H_n(B, A) & \longrightarrow & H_n(X, A) \\ & \nwarrow \scriptstyle [1] & \swarrow \\ & H_n(X, B) & \end{array}$$

from the SES (of quotient cpxs)

$$0 \rightarrow C_*(B, A) \longrightarrow C_*(X, A) \longrightarrow C_*(X, B) \longrightarrow 0$$

① When useful? Wait a bit; will use in conjunction w/
excision to show thm 2.13 finally
($\tilde{H}(X, A) \cong \tilde{H}(X/A)$ for good pair).

Excision



$$Z \subset A \subset X$$

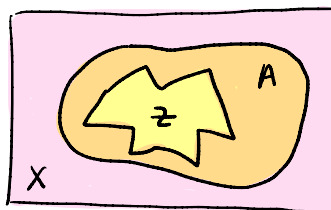
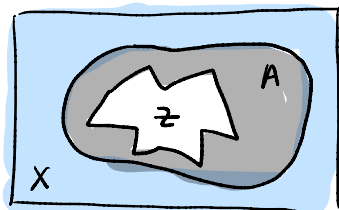
Want to excise (cut out) Z .

Thm 2.20 (Excision thm)

Given $Z \subset A \subset X$ where $\bar{Z} \subset \mathring{A}$, then
 $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isoms
 $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n.$

graded statement that I will write hereforth:

$$H_*(X-Z, A-Z) \xrightarrow{\cong} H_*(X, A)$$



$$\mathring{B} = X - \bar{Z} = \text{pink circle}$$

$$A = \text{yellow circle}$$

(color theory)

Equivalently, for subspaces $A, B \subset X$ st

$$\mathring{A} \cup \mathring{B} = X \quad \text{Interiors cover } X$$

the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isoms

$$H_*(B, A \cap B) \longrightarrow H_*(X, A).$$

recall defn (X, A) is a good pair if $\emptyset \neq A \subset X$, A closed, A is def. retract of some nbhd in X .
excision needed *help of equiv subpairs V, A of X .*

prop. 2.22 For good pairs (X, A) , the quotient map

$$q: (X, A) \longrightarrow (X/A, A/A)$$

induces isomorphisms

$$q_*: H_n(X, A) \longrightarrow H_n(X/A, A/A) \approx \tilde{H}_n(X/A).$$

on HW; this is how we go from relative to absolute homology groups

pf.

let V be a nbhd of A in X that def. retracts onto A .

(Maps below are the obvious induced ones from inclusion)

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\textcircled{1} \cong} & H_n(X, V) & \xleftarrow{\textcircled{3} \cong} & H_n(X-A, V-A) \\ \textcircled{6} \downarrow q_* & & \downarrow q_* & & \textcircled{5} \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\textcircled{2} \cong} & H_n(X/A, V/A) & \xleftarrow{\textcircled{4} \cong} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

show $\textcircled{1} - \textcircled{5}$ are isom. Since we have big square commutes, will get $\textcircled{6}$ is isom, which is what we want.

① & ②:

- $\tilde{H}_n(X, A) \xrightarrow{d_*} \tilde{H}_n(X, V)$ shows up in LES of triple buttressed by
 $\tilde{H}_n(V, A)$ and $\tilde{H}_{n-1}(V, A)$. (w/o those are 0.)

- The def. retract $V \rightsquigarrow A$ induces def. retract of pairs
 $(V, A) \rightsquigarrow (A, A)$.

- By homotopy equivalence of homology of a pair (prop. 2.19)

$$H_*(V, A) \cong H_*(A, A) \quad (\text{reduced or not})$$

- But the chains $C_*(A, A)$ are all 0.

$$\Rightarrow H_*(A, A) = 0. \Rightarrow H_*(V, A) = 0$$

- So $\tilde{H}_n(X, A) \xrightarrow{i_*} \tilde{H}_n(X, V)$ is an isom

③ & ④ These are $\cong_s$ by the Excision theorem!

⑤ At the chain level, the complexes can be identified because

$q|_{X-A}$ is a homeo on these spaces!

$\Rightarrow q_*$ is an isom on homology

$\Rightarrow$ since the squares commute, ⑥ must also be an isom.



$\mathcal{H}$

We will prove this version:

①

Equivalently, for subspaces $A, B \subset X$ st

$$\mathring{A} \cup \mathring{B} = X \quad \text{interiors cover } X$$

the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isom

$$H_*(B, A \cap B) \longrightarrow H_*(X, A).$$

Setup for pt of Exercise: then we need technical prop:
Space X .

$\mathcal{U} = \{U_j\}$ coll of spaces s.t. $\{U_j\}$ is an open cover of X .

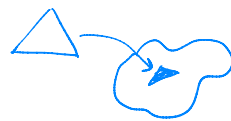
• Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$ be the subgroup gen'd by

σ_α where $\text{im}(\sigma_\alpha) \subset U_{j_\alpha}$ for some j_α .

(σ_α lands in one of the sets)

• (clearly) $\partial: C_n(X) \longrightarrow C_{n-1}(X)$ takes

$$C_n^{\mathcal{U}}(X) \longrightarrow C_{n-1}^{\mathcal{U}}(X).$$



$\Rightarrow \partial^2 = 0$, and the $C_n^{\mathcal{U}}(X)$ form a chain cpx.

$$\rightsquigarrow H_n^{\mathcal{U}}(X)$$

• WTS $H_n^{\mathcal{U}}(X) \cong H_n(X)$

chain htpy equivalence, $\Rightarrow$ isom on homology.

Main technical proposition

Step 1

prop. 2.21 The inclusion $\iota: C_n^u(X) \rightarrow C_n(X)$

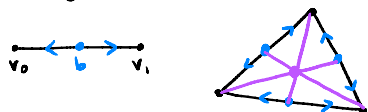
is a chain homotopy equivalence, i.e. $\exists f: C_n(X) \rightarrow C_n^u(X)$

$$\text{s.t. } \iota f \simeq \mathbb{1}_{C_n(X)}, \quad f \iota \simeq \mathbb{1}_{C_n^u(X)}$$

$$\Rightarrow \iota \text{ induces isom } H_n^u(X) \cong H_n(X).$$

pf.

(1) Barycentric subdivision



Fact diam of each resulting "smaller simplex" is

$$\leq \frac{n}{n+1} \text{ diam } \Delta^n$$

* size + simplicial comment

$\Rightarrow$ with r applications we can get simplices $\left(\frac{n}{n+1}\right)^r$
the original diam.

$$\frac{n}{n+1} < 1 \Rightarrow \lim \left(\frac{n}{n+1}\right)^r = 0 \quad \cup$$

Step 2

Want to construct a subdivision operator

$$S: C_n(X) \longrightarrow C_n(X), \text{ and show } S \simeq \mathbb{1}_{C(X)}$$

Then $S_* = \mathbb{1}_*$, i.e. subdividing doesn't change the homology.

(2) B. Sub. of linear chains

simpler setting

(only summary - used paper notes)

Consider $Y \subset \mathbb{R}^N$, convex.

- $\underline{LC}_n(Y) \subset C_n(Y)$ is gen'd by linear maps $\Delta^n \rightarrow Y$.

linear chains

- Work inductively using augmented complex, $\widetilde{LC}_*(Y)$
 $\widetilde{LC}_{-1}(Y) = \mathbb{Z}\langle [\emptyset] \rangle$, where for any 0-simplex $[w_0]$, $\partial[w_0] = [\emptyset]$.
- Identify linear chain $\lambda: \Delta^n \rightarrow Y$ w/ $[w_0, \dots, w_n]$
 where $w_i = \lambda(v_i)$.
- Define hom $b: LC_n(Y) \rightarrow LC_{n+1}(Y)$ for $b \in Y$.
 $[w_0, \dots, w_n] \mapsto [b, w_0, \dots, w_n]$

$$\rightsquigarrow \partial b + b \partial = \mathbb{1} (-0). \quad \textcircled{A}$$

Claim A: $\partial_{n+1} b_n = b_{n-1} \partial_n$

$$\partial b([w_0, \dots, w_n]) = \partial[b, w_0, \dots, w_n] = [w_0, \dots, w_n] - [b, \hat{w}_0, w_1, \dots, w_n] + \dots$$

$$= [w_0, \dots, w_n] - \sum_{i=0}^n (-1)^i [b, w_0, \dots, \hat{w}_i, \dots, w_n]$$

$$b \partial([w_0, \dots, w_n]) = \sum_{i=0}^n (-1)^i [b, w_0, \dots, \hat{w}_i, \dots, w_n] \Rightarrow \partial b + b \partial = \mathbb{1}.$$

//

- Use $b_\lambda = \text{barycenter of } [w_0, \dots, w_n]$ to define inductively subdivision operator $S(\lambda) = b_\lambda(S\partial\lambda)$ w/ $S_{-1}, S_0 = \mathbb{1}$. ⓑ

Claim ⓑ S is a chain map.

pf.

$$\begin{array}{ccccccc}
 LC_{n+1} & \longrightarrow & LC_n & \xrightarrow{\partial_n} & LC_{n-1} & \xrightarrow{\partial_{n-1}} & LC_{n-2} \\
 & & \downarrow S_n & \nearrow b_{n-1} & \downarrow S_{n-1} & \nearrow b_{n-2} & \downarrow S_{n-2} \\
 LC_{n+1} & \longrightarrow & LC_n & \xrightarrow{\partial_n} & LC_{n-1} & \xrightarrow{\partial_{n-1}} & LC_{n-2}
 \end{array}$$

Base Case

$$\begin{array}{ccc}
 LC_0 & \xrightarrow{\partial_0} & LC_{-1} \\
 S_0 \downarrow & \square & \downarrow S_{-1} \\
 LC_0 & \xrightarrow{\partial_0} & LC_{-1}
 \end{array}$$

$S_0, S_{-1} = \mathbb{1}$ satisfy

Assume $\partial_{n-1} S_{n-1} = S_{n-2} \partial_{n-1}$

Claim ⓐ

Recall defn: $S_n(\lambda) = b_{n-1} S_{n-1} \partial_n \lambda$

$$\partial_{n+1} b_n + b_{n-1} \partial_n = \mathbb{1}$$

WTS $\partial_n S_n = S_{n-1} \partial_n$, i.e. $\partial_n S_n - S_{n-1} \partial_n = 0$

$$\begin{aligned}
 \partial_n S_n - S_{n-1} \partial_n &= \partial_n (b_{n-1} S_{n-1} \partial_n) - S_{n-1} \partial_n = \underbrace{(\partial_n b_{n-1} - \mathbb{1})}_{\textcircled{a} - b_{n-2} \partial_{n-1}} (S_{n-1} \partial_n) \\
 &= -b_{n-2} \underbrace{\partial_{n-1} S_{n-1}}_{\text{ind. } S_{n-2} \partial_{n-1}} \partial_n = -b_{n-2} S_{n-2} \partial_{n-1} \partial_n = 0.
 \end{aligned}$$

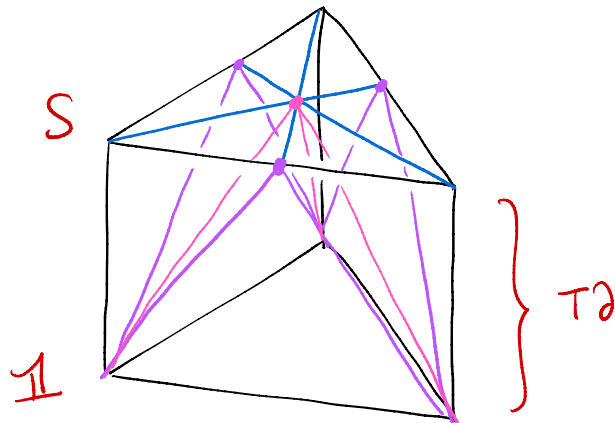
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- then define a chain htpy T b/w S and $\mathbb{I}$: inductively

$$T_{-1} = 0 \quad T\lambda = b_\lambda(\lambda - T\partial\lambda)$$

- show this is a chain htpy $T\partial + \partial T = \mathbb{I} - S$ (C)

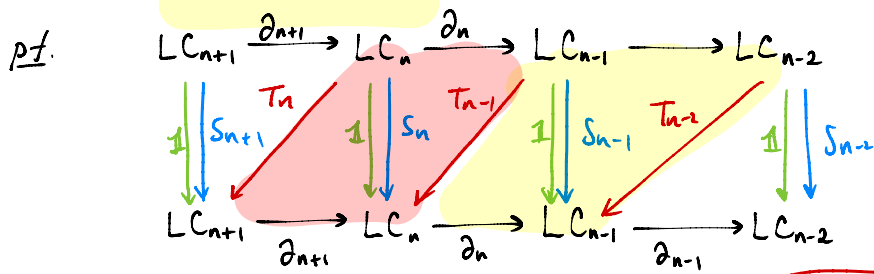
$\partial T =$ "boundary of T "



- Having built these maps, revert back to $LC_*(Y)$ (from $\widetilde{LC}_*$.)

- $\partial T + T\partial = \mathbb{I} - S$ still holds since $T_{-1} = 0$.

Claim C $T\partial + \partial T = 1 - S$



Assume the claim holds for $T_{-1}, \dots, T_{n-1}$, eg. $T_{n-2}\partial_{n-1} + \partial_n T_{n-1} = 1_{n-1} - S_{n-1}$

WTS $T_{n-1}\partial_n + \partial_{n+1}T_n = 1_n - S_n$

- $T_{n-1}\partial_n = b_{n-1}(1_{n-1} - T_{n-2}\partial_{n-1})\partial_n = b_{n-1}\partial_n$
- $\partial_{n+1}T_n = \partial_{n+1}(b_n(1_n - T_{n-1}\partial_n)) = \partial_{n+1}b_n - \partial_{n+1}b_n T_{n-1}\partial_n$
- $S_n = b_{n-1}(S_{n-1}\partial_n) = b_{n-1}(1_{n-1} - T_{n-2}\partial_{n-1} - \partial_n T_{n-1})\partial_n$
 $= b_{n-1}\partial_n - 0 - b_{n-1}\partial_n T_{n-1}\partial_n$

$$\Rightarrow T_{n-1}\partial_n + \partial_{n+1}T_n + S_n \stackrel{?}{=} 1_n$$

$$= b_{n-1}\partial_n + (b_{n-1}\partial_n + \partial_{n+1}b_n) - (\partial_{n+1}b_n + b_{n-1}\partial_n)T_{n-1}\partial_n$$

$$= 1_n + [b_{n-1}\partial_n - T_{n-1}\partial_n] = 1_n. \quad \text{⑤}$$

$$b_{n-1}(1_{n-1} - T_{n-2}\partial_{n-1})\partial_n$$

$$= b_{n-1}\partial_n$$

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Step 2

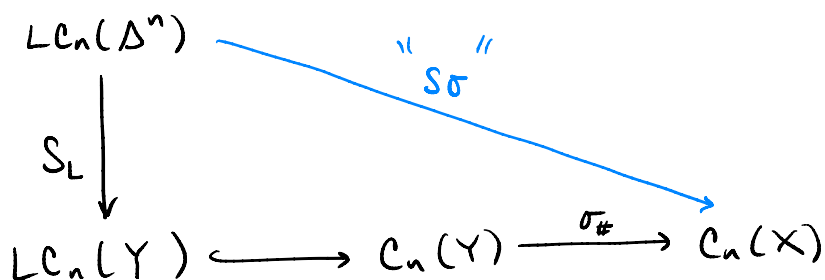
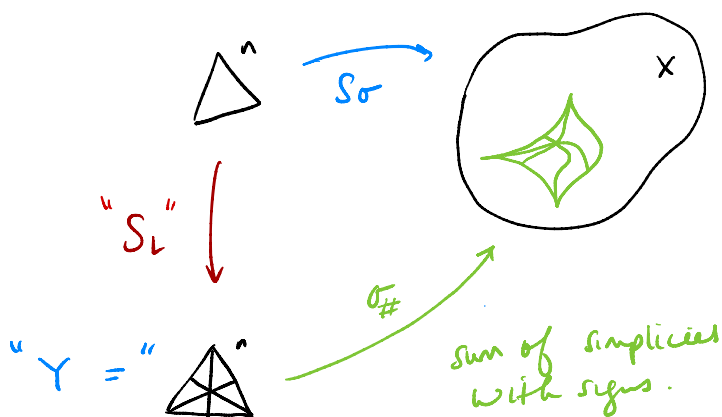
(3) Bary. subdiv of general chains

defn $S: C_n(X) \longrightarrow C_n(X)$

from S_L = the subdiv op from last time as

$$S\sigma = \sigma_{\#} S_L$$

view $\sigma_{\#}$ as chain map
induced by $\Delta^n \xrightarrow{\sigma} X$



WTS S is a chain map: Claim ① $\partial S = S \partial$ WTS $\partial S = S \partial$ (Conceptually not complicated)

$$\partial S \sigma = \partial \sigma_{\#} S_L \Delta^n = \sigma_{\#} \partial S_L \Delta^n = \sigma_{\#} S_L \partial \Delta^n$$

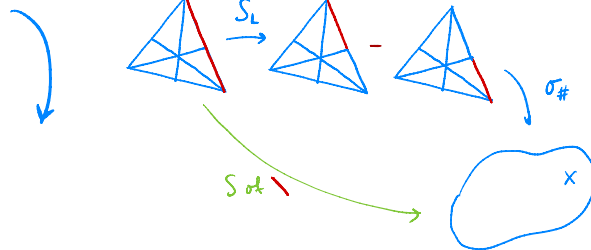
$\sigma_{\#}$ is chain map S_L is chain map

$$= \sigma_{\#} S_L \left(\sum_i (-1)^i \Delta_i^n \right)$$

Here Δ_i^n is the i^{th} facet (codim 1 face)

$$= \sum_i (-1)^i \sigma_{\#} S_L \Delta_i^n$$

$$= \sum_i (-1)^i S(\sigma|_{\Delta_i^n})$$



$$= S \left(\sum_i (-1)^i \sigma|_{\Delta_i^n} \right) = S \partial \sigma$$

Similarly define T from T_L :

$$T: C_n(X) \rightarrow C_{n+1}(X)$$

$$T \sigma = \sigma_{\#} T_L \text{id} \Delta^n$$

Check $\partial T + T \partial = \mathbb{1} - S$ still holds.

Same deal/conceptual picture.

Step 4

(4) Iterated.

(What if you need your simplices really small?)

Just need a chain htpy b/w $\mathbb{1}$ and S^m .

defn. operator $D_m = \sum_{0 \leq i < m} TS^i$

Check: $\partial D_m + D_m \partial$

$$= \sum_{i=0}^{m-1} \partial TS^i + TS^i \partial$$

$$= \sum \partial TS^i + T \partial S^i$$

$$= \sum (\mathbb{1} - S) S^i$$

$$= \sum_{i=0}^{m-1} S^i - S^{i+1}$$

$$= \mathbb{1} - S^m$$

$\downarrow$ S^i is chain map

$\downarrow$ T is already htpy

telescope!

Profit! (with some more work)

Recall $C_n^u(X)$

From analysis:

Open cover $\mathcal{U}$ of Δ^n , a compact metric space.

$\exists \varepsilon > 0$ (Lebesgue #) st. every set V of $\text{diam} < \varepsilon$ lies inside some $U \in \mathcal{U}$.

$\Rightarrow$ for each Δ^n we have some $\varepsilon(\Delta^n)$.

For each $\sigma: \Delta^n \rightarrow X$, defn. $m(\sigma) = \text{smallest } m \text{ s.t. } S^m \sigma$ is in $C_n^u(X)$. ← recall this?

We can now define $D: C_n(X) \rightarrow C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$ for each simplex σ .

Goal Chain map $f: C_n(X) \rightarrow C_n^u(X)$

that is a chain htpy inverse for $i: C_n^u(X) \rightarrow C_n(X)$.

Once we have this, we have chain htpy equiv b/w

$C_n^u(X)$ and $C_n(X)$.

(recall prop 2.24)

For each σ , we have $D\sigma = D_{m(\sigma)}\sigma$.

Let $f = 1 - \partial D - D\partial$ check chain map $\checkmark$.

Then $f(\sigma) = \sigma - \partial D\sigma - D\partial\sigma$

$$= \underbrace{\sigma - \partial D_{m(\sigma)}\sigma}_{\substack{\text{linear combo of} \\ \text{different } D_{m(\sigma)}\sigma}} - D\partial\sigma$$

linear combo of
different $D_{m(\sigma)}\sigma$

$$\begin{aligned} \partial D_m + D_m \partial &= 1 - S^m \\ \Rightarrow 1 - \partial D &= S^m - D\partial \end{aligned}$$

$$S^{m(\sigma)} + D_{m(\sigma)}(\partial\sigma)$$

$$= \underbrace{S^m \sigma}_{\substack{\text{linear combos of terms} \\ \text{in } C_n^u(X)}} + \underbrace{D_{m(\sigma)} \partial\sigma}_{\substack{\text{linear combos of terms} \\ \text{in } C_n^u(X)}} - D\partial\sigma$$

in $C_n^u(X)$

linear combos of terms

$$D_{m(\sigma)} \sigma_j - D_{m(\sigma_j)} \sigma_j$$

$$\text{where } m(\sigma_j) \leq m(\sigma)$$

smaller
facets!

biggin simplex!

$\Rightarrow$ all terms live in $C_n^u(X)$!

So f can be viewed as a map $C_n(X) \rightarrow C_n^u(X)$
(previously f was i) (restrict target)

Claim f, ι are htpy inverses

pt

- Now $\varphi = 1 - \partial D - D\partial \Leftrightarrow \partial D + D\partial = 1 - \varphi$

$$\Rightarrow \partial D + D\partial = 1 - \varphi$$

$$C^u(X) \hookrightarrow C(X) \xrightarrow{\quad} C(X) \xrightarrow{\quad} C^u(X)$$

- $\Rightarrow f\iota = 1$

//

end pt of prop. 2.21!

Pf. of excision thm (restatement) on paper.

Cover: $X = A \cup B$. Let's just say $A = A^\circ$, $B = B^\circ$... (open cover)

Write $C_n(A+B) := C_n^u(X)$ more intuitive notation

The htpy equations $\partial D + D\partial = 1 - 1\partial$, $p_! = 1$

all descend to the quotient by $C_*(A)$ (check mentally)

$$\Rightarrow \textcircled{1} \quad C_n(A+B)/C_n(A) \xrightarrow{\quad \bar{i} \quad} C_n(X)/C_n(A)$$

Induces isom on homology

$$\textcircled{2} \quad C_n(B)/C_n(A \cap B) \xrightarrow{\quad i \quad} C_n(A+B)/C_n(A)$$

can be an identification at chain level

$\Rightarrow$ together we have maps inducing isom on homologies:

$$H_n(B, A \cap B) \cong H_n(X, A).$$

Go home and take a deep breath. ☺