

New goal: CW homology: easiest way to compute H_* .

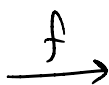
Recall Cell/CW complexes are built by inductively attaching k -cells $e_\alpha^k \cong D^k$ to the $(k-1)$ skeleton X^{k-1} , via attaching maps $\partial D^k = S^{k-1} \rightarrow S^{k-1} \subset X^{k-1}$.

Observe $f: S^n \rightarrow S^n$ ($n > 0$) induces

$$\begin{array}{ccc} f_*: H_n(S^n) & \rightarrow & H_n(S^n) \\ \text{"id"} \uparrow & & \uparrow \text{"id"} \\ \mathbb{Z} & \xrightarrow{d.} & \mathbb{Z} \\ \alpha & \mapsto & d \cdot \alpha \end{array}$$

The "d" here is called the degree of the map f b/w spheres of the same dim, denoted deg f .

eg.



crush all red to basepoint

gives a degree 3 map.

CW cpxs are entirely characterized through their attaching maps, which are maps $S^k \rightarrow S^k$.

Here's a list of "basic" properties for maps $f: S^n \rightarrow S^n$.

① $\deg \text{id} = 1$.

② $f = g \Rightarrow f_* = g_* \Rightarrow \deg f = \deg g$.

* Converse is harder! Cor. 4.25 in Chp. 4.
We might come back to this.

③ • $(fg)_* = f_* g_* \Rightarrow \deg(fg) = \deg f \cdot \deg g$.

• Also, if f is a hpy equiv then $\exists g$ st.

$fg \simeq \text{id} \Rightarrow f_* g_* = 1 \Rightarrow \deg f = \pm 1$.

④ f not surjective $\Rightarrow \deg f = 0$.

Choose $x_0 \in S^n - f(S^n)$.

f factors as

$$S^n \xrightarrow{f'} \underbrace{S^n - \{x_0\}}_{\cong D^n \cong *}$$

$\Rightarrow f'_* = 0 \Rightarrow f_* = i_* f'_* = 0$.

⑤ If f is a reflection map that fixes an equatorial S^{n-1} and exchanges the two hemispheres, then $\deg f = -1$.

Use the Δ -cpx structure and check

$[\Delta_1^+ - \Delta_2^+] \mapsto [\Delta_2^- - \Delta_1^-]$.



⑥ On the other hand, the antipodal map

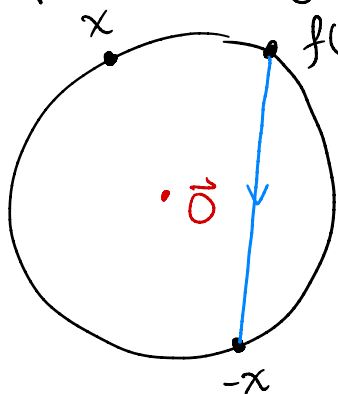
$-1: S^n \rightarrow S^n$ is the product of $(n+1)$ reflections
 $x \mapsto -x$

$\Rightarrow$ has degree $\deg(-1) = (-1)^{n+1}$.

⑦ Lemma If $f: S^n \rightarrow S^n$ has no fixed pts,
then $f \simeq -1$. (!!!)

Pf.

$\forall x \in S^n$, $f(x) \neq x$, so there is a line
from $f(x) \rightsquigarrow -x$ in D^{n+1} that doesn't
pass through the origin:



$t \mapsto \underbrace{(1-t)f(x) - tx}_{\text{nowhere } 0.}$
note the minus
 $+ t(-x)$
 $\uparrow$
 -1

So the function (a rescaling)

$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$ is continuous in
both t and x .

and is a htpy from f to -1 .

As a corollary, we get:

prop. 2.29 $\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group that can freely act on S^n if n is even.

Pf.

G acts by homeos, which have degree ± 1 .
 $G \curvearrowright S^n$ freely $\Rightarrow$ if $g \neq 1$, then the homeo (ϕ_g) has no fixed points.

$$\Rightarrow g \cong -1 \Rightarrow \deg g = (-1)^{n+1} = -1 \text{ since } n \text{ odd.}$$

We have a degree function

$$d: G \rightarrow \{\pm 1\}$$

that must be a homom since $\deg fg = \deg f \cdot \deg g$.

But kernel is trivial, so $G \subset \mathbb{Z}/2\mathbb{Z}$. $\square$

For application of degree: hairy Ball Theorem

(Usual proof - by index vs. χ ?)

thm 2.28 S^n has a continuous field of nonzero tangent vectors iff n is odd,

i.e. TS^n has a section iff n is odd.

we might discuss VBS later.

Pf.

$$\boxed{\Leftarrow} \quad S^{2n-1} \subset \mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}$$

$$\text{Let } X = \sum_{i=1}^n -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$$

(How (?) show this is nonvanishing VF on sphere.)

$\Rightarrow$ Suppose $x \mapsto v(x)$ is a nonvanishing VF.

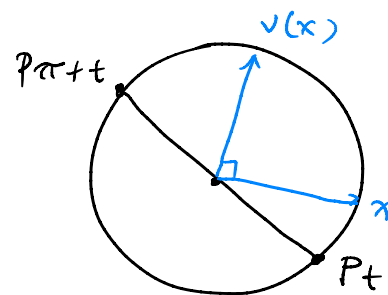
$\Rightarrow x \perp v(x)$ in $\mathbb{R}^{n+1}$ (tangency condition)

Since $\text{def } v(x) \neq 0$, normalize so that $\|v(x)\| = 1$,
so we can view $v(x) \in S^n$.

Now consider the vectors

$$p_t = (\cos t)x + (\sin t)v(x) \in S^n$$

Note on $t \in [0, \pi]$,
we go from $x \rightsquigarrow -x$.



$$\Rightarrow \mathbb{1} \simeq -\mathbb{1}$$

$$\Rightarrow \deg -\mathbb{1} = 1 \Rightarrow n \text{ is odd.}$$

$\square$

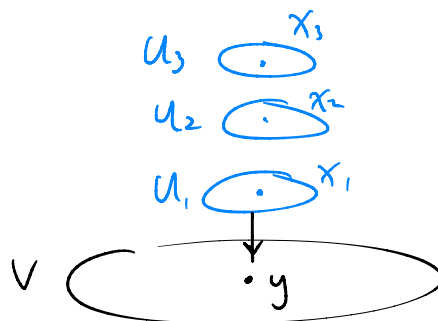
How to compute degree: local degrees

Suppose $f: S^n_{\text{src}} \rightarrow S^n_{\text{tar}}$ $n > 0$ with the property that for $y \in S^n_{\text{tar}}$,

$$f^{-1}(y) = \text{finitely many points} \\ x_1, \dots, x_m$$

Then we can choose nbhds U_i of x_i s.t. and $f(U_i) \subset V$, a fixed nbhd of $f(x_i) = y$.

Pancakes on plate:



Then $f(U_i - x_i) \subset V - y$ for each i .

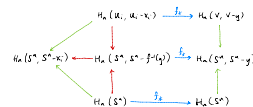
defn. The local degree of f at x_i ,

denoted $\deg f|_{x_i}$, is the "degree" of the induced homomorphism

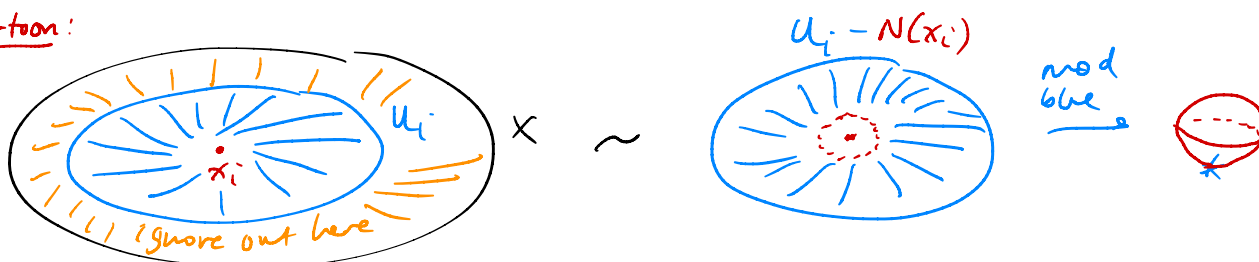
$$H_n(U_i, U_i - x_i) \xrightarrow{f_*} H_n(V, V - y)$$

need to define carefully

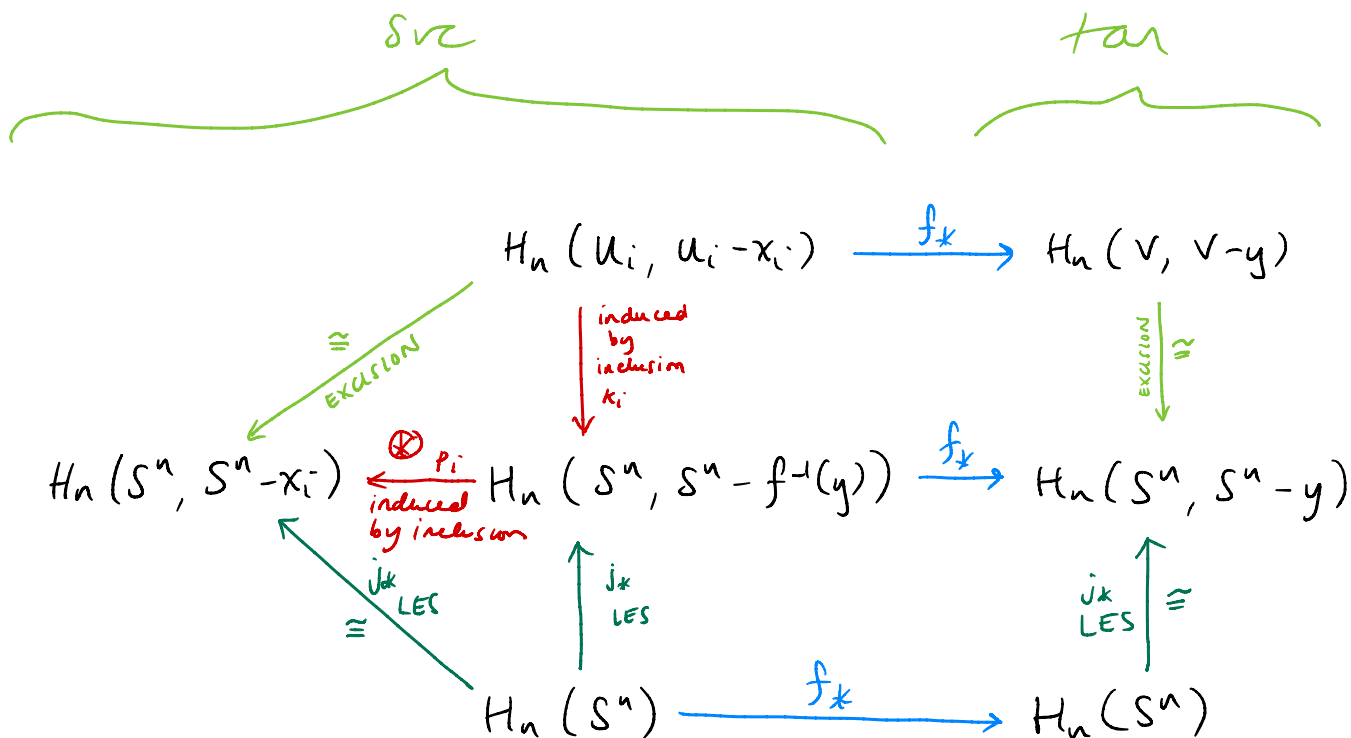
mls indeed, $H_n(U_i, U_i - x_i), H_n(V, V - y) \cong \mathbb{Z}$:



Cartoon:

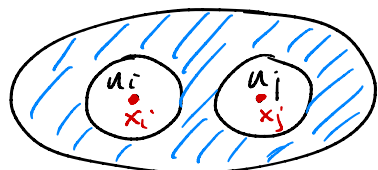


We claim the following diagram commutes.



• Check all answers other than red ones make sense.

⊕ By Excision, $H_n(S^n, S^n - f^{-1}(y)) \cong \bigoplus_i H_n(U_i, U_i - x_i)$:



excise the blue. $\uparrow$ $\perp$ of $(U_i, U_i - x_i)$
(x_i, x_j are missing)

Now consider the triangles and squares.

$$\begin{array}{ccccc}
 H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V - y) \\
 \swarrow \text{(inclusion)} \cong & \downarrow \text{composition } \circ & \searrow \text{excision} \cong & \text{naturality} & \\
 H_n(S^n, S^n - x_i) & \xleftarrow{p_i} & H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n - y) \\
 \swarrow \text{composition } \circ & \uparrow \text{LES } j_* & \text{naturality} & \uparrow \text{LES } j_* & \\
 H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) & & H_n(S^n) \\
 \cong \mathbb{Z} & & \cong \mathbb{Z} & & \cong \mathbb{Z}
 \end{array}$$

Now we can use the green arrows on the left & right sides to identify the src & tar groups of

$$H_n(U_i, U_i - x_i) \xrightarrow{f_*} H_n(V, V - y)$$

with $\mathbb{Z}$. (b/c of bottom row).

prop. 2.30 (with the assumption $f^{-1}(y)$ finite)

$$\deg f = \sum_i \deg f|_{x_i} \quad (\text{any choice of } y!)$$

pf.

$$\begin{array}{ccccc}
 H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V - y) \\
 \swarrow \cong & \downarrow \text{composition } \circ & \searrow \cong & & \\
 H_n(S^n, S^n - x_i) & \xleftarrow{p_i} & H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n - y) \\
 \swarrow \cong & \uparrow \text{LES } j_* & \text{naturality} & \uparrow \text{LES } j_* & \\
 H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) & & H_n(S^n)
 \end{array}$$

(Simplified the notation - renamed the maps)

- Since $1 \xleftarrow{p_i} \textcircled{?}$ commutes for all i ,
 $(p_i \circ j)(1) = 1 \quad \forall i$

we have $j(1) = (1, \dots, 1) = \sum_i k_i(1)$

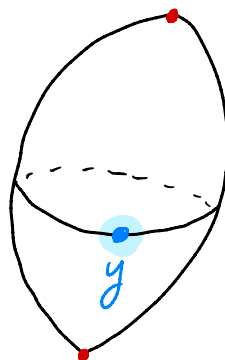
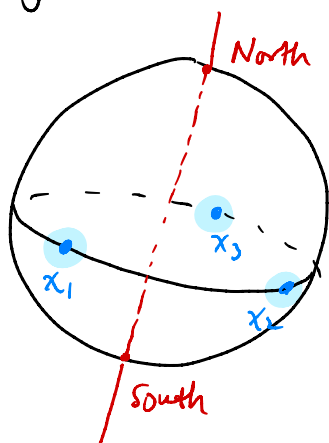
- upper square commutes

$$\begin{array}{ccc}
 H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V - y) \\
 \downarrow \scriptstyle K_i & \downarrow \scriptstyle 1 & \downarrow \scriptstyle \approx \\
 H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n - y) \\
 (0, \dots, \underset{\substack{\uparrow \\ \text{ith}}}{1}, \dots, 0) & \longmapsto & \deg f|_{x_i} \cdot 1 \\
 \Rightarrow (1, 1, \dots, 1) & \longmapsto & \sum_i \deg f|_{x_i} \cdot 1 \\
 & & \uparrow \\
 & & \deg f \cdot 1
 \end{array}$$

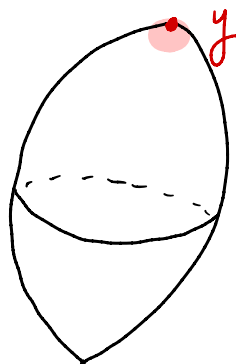
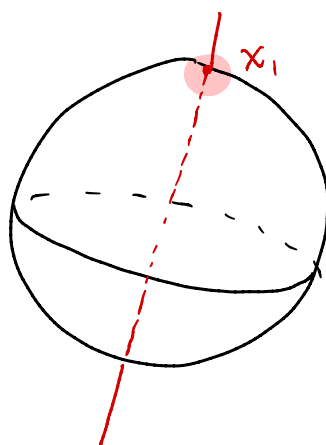
$$\Rightarrow \deg f = \sum_i \deg f|_{x_i}$$

□

eg. $f: S^2 \rightarrow S^2$ rotation by $2\pi/3$



$$\deg f = 1 + 1 + 1.$$



$$\deg f = 3$$

↑
local
degree

At x_1 , the map looks like $z \mapsto z^3$ (in $\mathbb{C}$).

rmk In general can get a degree d map $S^n \rightarrow S^n$
by suspending $f: S^1 \rightarrow S^1$ ($S^1 \subset \mathbb{C}$)
 $z \mapsto z^d$

(For negative k , precompose with orientation reversal.)

A few facts about singular hom. of CW cpxs:

Lemma 2.34 If X is a CW cpx, then

(a) Say X^n is obtained by attaching $\{e_\alpha^n\}_{\alpha \in A}$ to X^{n-1} .

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & \text{for } k \neq n \\ \bigoplus \mathbb{Z} \langle e_\alpha^n \rangle & \end{cases}$$

(X^n, X^{n-1}) is a good pair, and $X^n/X^{n-1} \cong \bigvee_{\alpha} S^n$

(b) $H_k(X^n) = 0$ for $k > n$.

↳ If X is finite diml then

$H_k(X) = 0$ when $k > \dim X$.

(c) The inclusion $i: X^n \hookrightarrow X$ induces

an isom

$$i_*: H_k(X^n) \longrightarrow H_k(X) \quad \text{if } k < n.$$

k^{th} homology only depends on the X^{k+1} skeleton
again makes sense thinking about simplicial hom.

pf. (a) done

Consider LES for pair (X^n, X^{n-1}) ,

$$\underbrace{H_{k+1}(X^n, X^{n-1})}_{=0 \text{ if } k \neq n-1} \rightarrow H_k(X^{n-1}) \rightarrow H_k(X^n) \rightarrow \underbrace{H_k(X^n, X^{n-1})}_{=0 \text{ if } k=n}$$

- if $k > n$ then we have

$$H_k(X^n) \cong H_k(X^{n-1}) \cong \dots \cong H_k(X^0) = 0.$$

This proves (b).

- if $k < n$, then

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \dots \cong H_k(X^{nm}).$$

$\hookrightarrow$ so if $X = X^d$, then this proves (c)

pt of (c) for infinite dim case (sketch/discussion)

- again need to use the fact that singular chains have cpt image and thus meets only finitely many cells (last time it was simplices for H^A)
- so each chain lives in a finite skeleton X^m
- ...

The cellular chain $C_p X$

$X = CW\text{ cpx}$, $X^n = n\text{-skeleton}$.

recall we have LES

$$\hookrightarrow H_n(X^{n-1}) \xrightarrow{i_*} H_n(X^n) \xrightarrow{j_*} H_n(X^n, X^{n-1})$$

where $\partial[\alpha] = [\partial\alpha]$.

we define the cellular chain $C_p X$

$(C_*^{CW}(X), d_*)$ with $C_n(X) := H_n(X^n, X^{n-1})$ and d_n by

$$\begin{array}{ccccccc} \dots & \rightarrow & H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) \rightarrow \dots \\ & & \uparrow \partial_{n+1} & & \downarrow j_{*,n} & & \uparrow \dots \\ & & H_n(X^n) & & & & \\ & & & & \downarrow \partial_n & & \uparrow j_{*,n-1} \\ & & & & H_{n-1}(X^{n-1}) & & \end{array}$$

(weave together LESs)

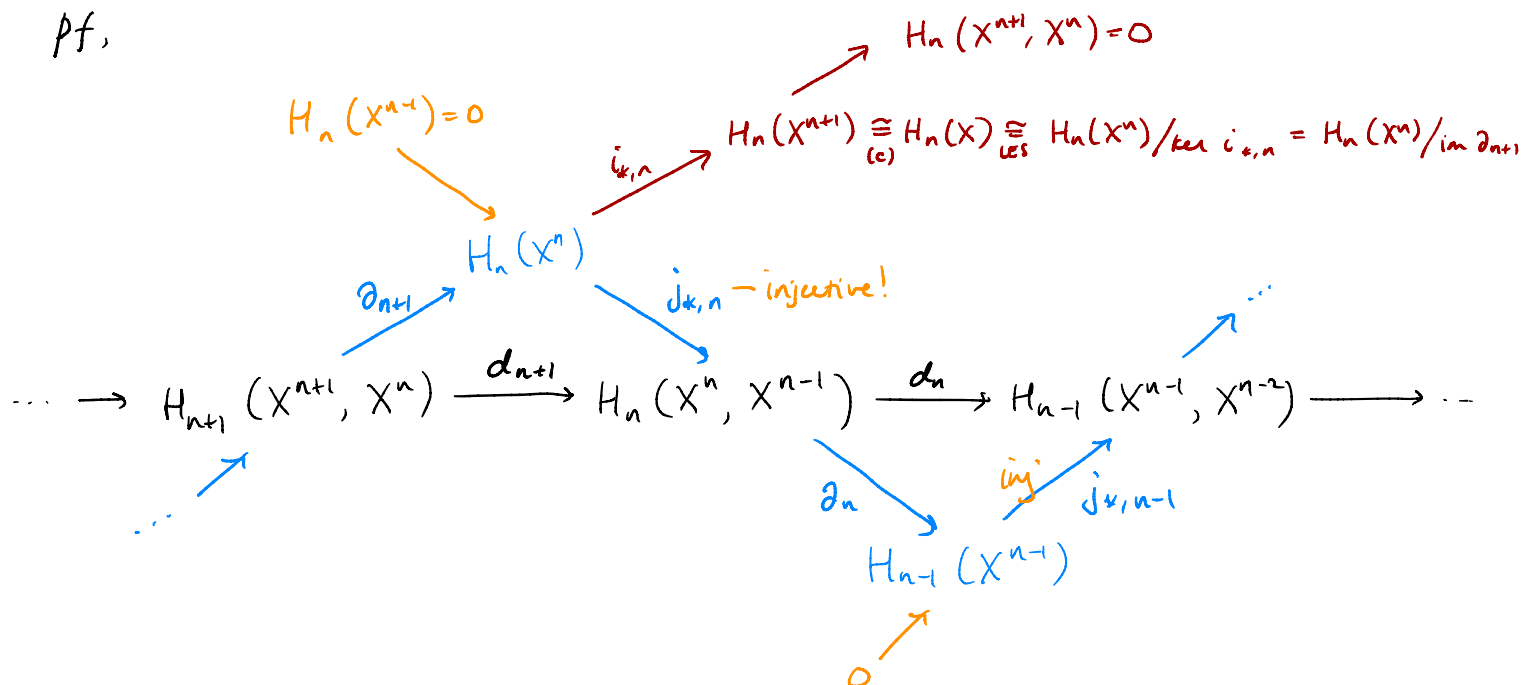
Check: indeed $d^2 = 0$:

$$d_n \circ d_{n+1} = j_{*,n-1} \circ \underbrace{\partial_n \circ j_{*,n}}_{\text{consecutive maps in LES}} \circ \partial_{n+1}$$

consecutive maps in LES $\Rightarrow = 0$.

thm. $H_*^{cw}(X) \cong H_*(X)$ (for all $\text{qpx } X$).

Pf.



- $j_{k,n}$ injective
 $\Rightarrow j_{k,n}$ takes $\text{im } \partial_{n+1}$ isomorphically onto $\text{im } d_{n+1}$
 $\Rightarrow \text{im } \partial_{n+1} \cong \text{im } d_{n+1}$
- $j_{k,n}$ injective $\Rightarrow$ takes $H_n(X^n)$ isomorphically onto $\underline{\ker \partial_n} \subset H_n(X^n, X^{n-1})$
- $j_{k,n-1}$ injective $\Rightarrow \ker d_n = \underline{\ker \partial_n}$.
 $\Rightarrow H_n(X^n) \cong \ker d_n$
- $\Rightarrow H_n(X) \cong H_n(X^n) / \text{im } \partial_{n+1} \cong \ker d_n / \text{im } d_{n+1}$



Next: How to practically compute $H_{\text{CW}} = H_{\text{sing}}$

How to actually compute d_n ?

F 2.6.26

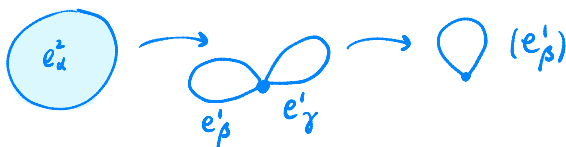
Cellular Boundary Formula. $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$ where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \rightarrow X^{n-1} \rightarrow S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

(pf will basically be omitted)

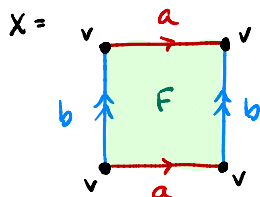
$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

degree of map

$$S_\alpha^{n-1} \rightarrow X^{n-1} \rightarrow S_\beta^{n-1}$$



eg. Torus



$$d_{Fa} = 1 - 1 = 0$$

$$d_{Fb} = 1 - 1 = 0$$

Whole calculation: Summary $d=0$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}\langle F \rangle & \longrightarrow & \mathbb{Z}\langle a, b \rangle & \longrightarrow & \mathbb{Z}\langle v \rangle \longrightarrow 0 \\ & & F \longmapsto 0 & & a \longmapsto 0 & & \\ & & & & b \longmapsto 0 & & \end{array}$$

$$H_n(X) = \begin{cases} \mathbb{Z} & n=0, 2 \\ \mathbb{Z}^2 & n=1 \\ 0 & \text{o/w} \end{cases}$$

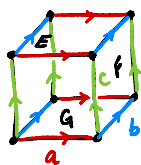
Q. Another space with same homology? $\rightsquigarrow$ Moore spaces

In fact, by similar calculation we get

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^{2g} & n=1 \\ 0 & \text{o/w.} \end{cases}$$

==

eg. $T^3 = S^1 \times S^1 \times S^1$



- all vertices are v
- all parallel edges are the same edge: a, b, c
- all parallel faces are the same face E, F, G

$$\partial E = aba^{-1}b^{-1}, \partial F = bcb^{-1}c^{-1}, \partial G = acc^{-1}a^{-1} \leftarrow \text{topological statement.}$$

$$\Rightarrow d_2(E), d_2(F), d_2(G) \text{ are all } 0!$$

- Single 3-cell filling in the cube.

$$d(e^3) = E - E + F - F + G - G$$

$$H_n(T^3) \cong \begin{cases} \mathbb{Z} & n=0,3 \\ \mathbb{Z}^3 & n=1,2 \\ 0 & \text{o/w} \end{cases}$$

(notice a pattern?) $\rightsquigarrow$ Künneth formula

HW: $\mathbb{R}P^n$, using what you know about antipodal maps + degree from last week (more explicitly than in Hatcher)

$\hookrightarrow$ described $\mathbb{R}P^n, \mathbb{C}P^n$ in words.

Good example to have in pocket In odd dimensions: lens spaces

(3D) let p, q be coprime integers. we may assume $p > 0$.

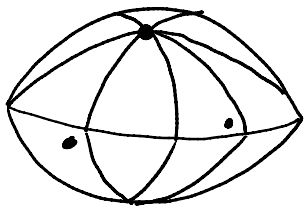
Consider $S^3 =$ unit sphere in $\mathbb{C}^2$.

Then $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$

is a generator of a free $\mathbb{Z}/p\mathbb{Z}$ rotation action.

defn. The lens space $L(p, q)$ is S^3/τ .

We can build $L(p, q)$ as follows:



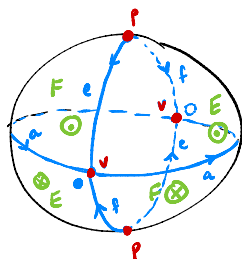
Cut the top and bottom hemispheres into p wedges.

Glue the Δ shown in the S hemisphere to the Δ on the N hemisphere that is a q/p turn away.

eg. (More complicated cell structure than necessary)

$L(2,1) = \mathbb{R}P^3$: Check that this gives the antipodal map.

Build a CW complex:



$$d_0 = 0$$

$$d_1(a) = 0 \quad d_1(e) = v - p \quad d_1(f) = v - p$$

$$d_2(E) = a - f + e \quad d_2(F) = a - e + f$$

$$d(e^3) = 0.$$

$$\begin{aligned} \ker d_1 &= \mathbb{Z} \langle a, e-f \rangle \\ &= \mathbb{Z} \langle a, a+e-f \rangle \end{aligned}$$

$$\begin{aligned} \operatorname{im} d_2 &= \mathbb{Z} \langle a-f+e, a-e+f \rangle \\ &= \mathbb{Z} \langle 2a, a+e-f \rangle \end{aligned}$$

$$H_1 \cong \mathbb{Z}/2\mathbb{Z}$$

$$\ker d_2 = 0, \operatorname{im} = 0 \Rightarrow H_1 \cong 0$$

$$H_3 \cong \mathbb{Z}.$$

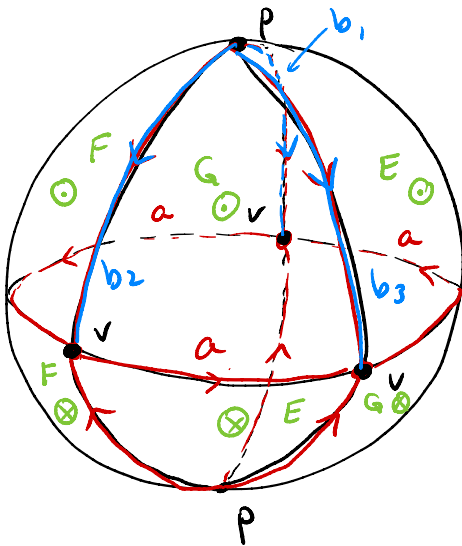
or

In general?

$$H_n(L(p,q)) = \begin{cases} \mathbb{Z} & n=0,3 \\ \mathbb{Z}/p\mathbb{Z} & n=1 \\ 0 & \text{o/w} \end{cases}$$

* However, $L(5,2) \neq L(5,1)$!

(Alexander 1919, ^{diff}htpy type)



Compute H_1 :

$$d_1(a) = 0$$

$$d_1(b_i) = v - p$$

$$d_2(E) = a - b_1 + b_3$$

$$d_2(F) = a - b_2 + b_1$$

$$d_2(G) = a - b_3 + b_2$$

$$\begin{aligned} \ker d_1 &= \langle a, b_1, -b_2, b_2 - b_3, \cancel{b_3 - b_2} \rangle \\ &= \langle a, \underline{a - b_2 + b_1}, \underline{a - b_2 + b_3} \rangle \end{aligned}$$

$$\begin{aligned} \operatorname{im} d_2 &= \langle a - b_1 + b_3, a - b_2 + b_1, a - b_3 + b_2 \rangle \\ &= \langle \underline{a - b_1 + b_3}, \underline{a - b_2 + b_1}, 3a \rangle \end{aligned}$$