M 2.9.26

# Mayer-Vietoris Sequences

↳ compute $H_*$ via gluing open sets.

**thm.**

Say $A, B \subset X$ s.t. $X = \mathring{A} \cup \mathring{B}$

WLOG may assume $A, B$ open.

$\mathcal{U} = \{A, B\}$ open cover.

There is a LES

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots \quad \cdots \to H_0(X) \to 0.$$

induced by the SES

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0$$

$$x \longmapsto (x, -x)$$

$$(x, y) \longmapsto x + y.$$

and $\Phi = \varphi_*$, $\Psi = \psi_*$ in the LES.

Rmks ① "$C_n(A+B)$" $= C_n^{\mathcal{U}} \subset C_n(X)$ is actually $C_n(A) \oplus C_n(B)$ as in the pf of excision.

Recall that $C_\bullet(A+B) \hookrightarrow C_\bullet(X)$ induces $\cong$ on homology.

② <u>what is the connecting map $\partial$?</u>

$$\partial : H_n(X) \longrightarrow H_{n-1}(A \cap B)$$
$$[z] \longmapsto \partial[z]$$

cycle $z \xrightarrow[\text{subdivide}]{\text{barycentric}} z = x + y$ with $x \in C_n(A)$, $y \in C_n(B)$.

* note $x, y$ are just chains, not necessarily cycles.

$\partial z = 0 \Rightarrow \partial(x+y) = 0 \Rightarrow \partial x = -\partial y$.

$$\begin{tikzcd}
 & C_{n-1}(A \cap B) \arrow[d, "\varphi"'] \\
C_n(A) \oplus C_n(B) \arrow[r, "\partial"] \arrow[d, "\psi"'] & C_{n-1}(A) \oplus C_{n-1}(B) \\
C_n(A+B) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \partial x \arrow[d, mapsto] \\
(x, y) \arrow[r, mapsto] \arrow[d, mapsto] & (\partial x, \partial y) = (\partial x, -\partial x) \\
z = x + y &
\end{tikzcd}$$

So $\partial[z] = [\partial x] = [-\partial y]$.

is a bdy $\Rightarrow$ is a cycle.

③ There is an analogous M-V seqn for reduced homology, where we augment the SES by

$$\begin{tikzcd}
0 \arrow[r] & C_0(A\cap B) \arrow[r, "\varphi"] \arrow[d, "\varepsilon"] & C_0(A)\oplus C_0(B) \arrow[r, "\psi"] \arrow[d, "\varepsilon\oplus\varepsilon"] & C_0(A+B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$a \longmapsto (a, -a)$$

$$(b, c) \longmapsto b+c.$$

Same story as usual.

④ We'll see that $H_1(X) =$ abelianization of $\pi_1(X)$ for $X$ path-cntd.

Suppose $X$, $A\cap B$ are path cntd.
Then $\tilde{H}_0(A\cap B) = 0$ so

$$H_1(X) \cong H_1(A)\oplus H_1(B) \Big/ \ker \Psi = \operatorname{im}\Phi.$$

$$\operatorname{im}\Phi = \langle \text{chains of the form } (x, -x) \rangle$$

This is the abelianized statement of Seifert-van Kampen!

⑤ more generally, if $X = A \cup B$ where
$A, B$ are def retracts of nbhds $U, V$ (resp)
with $U \cap V$ def retracting onto $A \cap B$,

$(U, V, U \cap V)$ $(U, V$ open nbhds$)$

$\downarrow$ def retract

$(A, B, A \cap B)$

then 5-lemma implies we have red isomorphism;

$$\begin{array}{ccccccccc}
H_n(A\cap B) & \longrightarrow & H_n(A)\oplus H_n(B) & \longrightarrow & H_n(A+B) & \longrightarrow & H_{n-1}(A\cap B) & \longrightarrow & H_{n-1}(A)\oplus H_{n-1}(B) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H_n(U\cap V) & \longrightarrow & H_n(U)\oplus H_n(V) & \longrightarrow & H_n(U+V) & \longrightarrow & H_{n-1}(U\cap V) & \longrightarrow & H_{n-1}(U)\oplus H_{n-1}(V) \\
 & & & & \downarrow\cong & & & & \\
 & & & & H_n(X) & & & &
\end{array}$$

Hence we can use M-V with this more general cover $A \cup B$.

↳ In particular, for CW cpx, can use cell rather than nbhds of cells!

eg. M-V is useful for induction arguments:

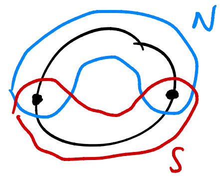
$X = S^n$, $\quad A$ = Northern Hemisphere (with equator)

$\qquad\quad B$ = Southern ——————

$\qquad\quad A \cap B$ = equator $\cong S^{n-1}$.

$A, B \cong D^n \Rightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) = 0 \quad \forall i$.

$\Rightarrow$ we get isomorphisms $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$.

sub eg $\quad S^1 =$  $S^0$ $\quad$ ($S^0$ not path cntd)

$\hookrightarrow \tilde{H}_1(S^0) = 0 \longrightarrow \underset{0 \,\oplus\, 0}{\tilde{H}_1(D^1) \oplus \tilde{H}_1(D^1)} \longrightarrow \underset{?}{\tilde{H}_1(S^1)} \hookleftarrow$

$\hookrightarrow \tilde{H}_0(S^0) = \mathbb{Z} \longrightarrow 0$

$\rightsquigarrow$ no path connectedness conditions needed for M-V ☺.

eg. Klein bottle (as cpx)

$K = \underset{A \,\cup\, B}{M \cup M}$ $\quad$ union of 2 Möbius bands

$M$ retracts onto its core circle $\gamma$. $\quad A \cap B = S^1$

Reduced M-V: $\quad 0 \longrightarrow \underset{?}{H_2(K)} \longrightarrow$

$\longrightarrow \underset{\mathbb{Z}}{H_1(A \cap B)} \xrightarrow{\Phi} \underset{\mathbb{Z} \,\oplus\, \mathbb{Z}}{H_1(A) \oplus H_1(B)} \xrightarrow{\Psi} \underset{?}{H_1(K)} \longrightarrow 0$

$\partial M \longmapsto (2\gamma, -2\gamma)$

injective! $\qquad \Rightarrow H_2(K) \cong 0$.

$\Psi$ surjective $\Rightarrow H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

use basis: $(1,0)$, $(1,-1)$

//

## Euler characteristic (shadow of homology)

Many different interpretations (eg. w/ vector fields)

for us:

defn For a CW cpx X, $\chi(X) = \#\text{0-cells} - \#\text{1-cells} + \#\text{2cells} - \dots$

prop. $\chi(X)$ doesn't depend on the CW complex structure.

In fact, $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rk} H_n(X)$

This is a purely algebraic fact + works for all chain cpxs of abelian groups. ($\mathbb{Z}$-modules)

<u>lemma</u>. If $0 \to A \to B \to C \to 0$ is a SES of finitely generated $\mathbb{Z}$-mods then rank $B$ = rank $A$ + rank $C$.

$C \cong B/A$, rank $C$ = rank $B$ - rank $A$.

<u>thm</u>. $\chi(\mathcal{C}_\bullet) = \chi(H_\bullet(\mathcal{C}))$ $(\mathcal{C}, d)$ a chain cpx.

pf.

$Z_n$ = cycles in $C_n$, $B_n$ = bdys in $C_n$, $H_n = Z_n/B_n$

① $0 \to B_n \to Z_n \to H_n \to 0$

$\Rightarrow \text{rk } Z_n = \text{rk } B_n + \text{rk } H_n$

How to relate to $C_n$? $C_n$ surjects onto its image under $d$.

② $0 \to \underbrace{Z_n}_{\ker d_n} \to C_n \xrightarrow{d_n} \underbrace{B_{n-1}}_{\text{im } d_n} \to 0$

$\Rightarrow \text{rk } C_n = \text{rk } B_{n-1} + \text{rk } Z_n$

$\Rightarrow \text{rk } C_n = \text{rk } H_n + \text{rk } B_n + \text{rk } B_{n-1}$

In alt sum, the $\text{rk } B_n + \text{rk } B_{n-1}$ cancels out. //

eg. $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$

eg. $\chi(N_\gamma) = 1 - \gamma + 1 = 2 - \gamma$

nonorientable

W 2.12.26

## An Algebraic Fact: Künneth Formula

dfn. let $\mathcal{A} = \cdots \xrightarrow{\partial_{n+1}^A} A_n \xrightarrow{\partial_n^A} A_{n-1} \xrightarrow{\partial_{n-1}^A} \cdots$

and $\mathcal{B} = \cdots \xrightarrow{\partial_{n+1}^B} B_n \xrightarrow{\partial_n^B} B_{n-1} \xrightarrow{\partial_{n-1}^B} \cdots$

be chain complexes.

The <u>bicomplex</u> $\mathcal{A} \otimes \mathcal{B}$ is

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \otimes B_{n+1} \arrow[r, "\partial \otimes 1"] \arrow[d, "1 \otimes \partial"] & A_n \otimes B_{n+1} \arrow[r, "\partial \otimes 1"] \arrow[d, "1 \otimes \partial"] & A_{n-1} \otimes B_{n+1} \arrow[r] \arrow[d, "1 \otimes \partial"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \otimes B_n \arrow[r, "\partial \otimes 1"] \arrow[d, "1 \otimes \partial"] & A_n \otimes B_n \arrow[r, "\partial \otimes 1"] \arrow[d, "1 \otimes \partial"] & A_{n-1} \otimes B_n \arrow[r] \arrow[d, "1 \otimes \partial"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \otimes B_{n-1} \arrow[r, "\partial \otimes 1"] \arrow[d] & A_n \otimes B_{n-1} \arrow[r, "\partial \otimes 1"] \arrow[d] & A_{n-2} \otimes B_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Observe that the red and blue maps <u>commute</u>.

The <u>totalization</u> of $\mathcal{A} \otimes \mathcal{B}$ is the flattening of $\mathcal{A} \otimes \mathcal{B}$ into a chain complex. To do this, we need to first make each square <u>anticommute</u> by negating every other column of blue differentials

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & A_{n+1}\otimes B_{n+1} \arrow[r, "\partial\otimes 1"] \arrow[d, "1\otimes\partial\cdot(-1)^{n+1}"] & A_n\otimes B_{n+1} \arrow[r, "\partial\otimes 1"] \arrow[d, "1\otimes\partial\cdot(-1)^{n}"] & A_{n-1}\otimes B_{n+1} \arrow[r] \arrow[d, "1\otimes\partial\cdot(-1)^{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1}\otimes B_{n} \arrow[r, "\partial\otimes 1"] \arrow[d, "1\otimes\partial\cdot(-1)^{n+1}"] & A_n\otimes B_{n} \arrow[r, "\partial\otimes 1"] \arrow[d, "1\otimes\partial\cdot(-1)^{n}"] & A_{n-1}\otimes B_{n} \arrow[r] \arrow[d, "1\otimes\partial\cdot(-1)^{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1}\otimes B_{n-1} \arrow[r, "\partial\otimes 1"] \arrow[d] & A_n\otimes B_{n-1} \arrow[r, "\partial\otimes 1"] \arrow[d] & A_{n-2}\otimes B_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

hom degree $n+n-1=2n-1$.

The totalization is defined to be

$$\mathrm{Tot}(\mathcal{A}\otimes\mathcal{B}) = \Big(\bigoplus_{i+j=n} A_i\otimes B_j\ ,\ \partial^{\mathrm{Tot}}\Big)$$

where for $a\in A_i$, $b\in B_j$, (Leibniz rule!)

$$\partial^{\mathrm{Tot}}(a\otimes b) = \partial_i^A a\otimes b + (-1)^i(a\otimes\partial_j^B b)$$

Sometimes people write "$\mathcal{A}\otimes\mathcal{B}$" for $\mathrm{Tot}(\mathcal{A}\otimes\mathcal{B})$.

Check: Because the squares anticommute, $(\partial^{\mathrm{Tot}})^2=0$ indeed.

Rmks

Some boundedness is needed for finite sums in $\partial^{\text{Tot}}$.

- If each chain group (orange diagonal) is a finite sum, then we are fine.
- eg. If working with the singular chain complex, we only have $A_i, B_j \neq 0$ when $i, j \geq 0$. Then each chain group is a finite sum.

<u>thm.</u> ("Künneth Formula") - general algebraic version.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are free

(ie all $A_i, B_j$ are free $\mathbb{Z}$-modules, as is the case for the singular chain cpx.)

Then there are canonically defined SESs th:

$$0 \to \bigoplus_{i+j=n} H_i(\mathcal{A}) \otimes H_j(\mathcal{B}) \to H_n(\mathcal{A} \otimes \mathcal{B})$$

$$\to \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(\mathcal{A}), H_j(\mathcal{B})) \to 0$$

which yield (noncanonical) isomorphisms

$$H_n(\mathcal{A} \otimes \mathcal{B}) \cong \bigoplus_{i+j=n} H_i(\mathcal{A}) \otimes H_j(\mathcal{B}) \;\oplus\; \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(\mathcal{A}), H_j(\mathcal{B}))$$

Rmks

① There is a proof in <u>Homotopical Topology</u> by Fomenko-Fuchs, for example.

② The fact that we have the SES does not immediately imply the "direct sum" statement.

We say a SES splits if there exists a "section" $s$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$

such that $\beta \circ s = id_C$.

There are many more equivalent characterizations of split SESs. (math 250)

Fact: If $0 \to A \to B \to C \to 0$ splits, then
$$B \cong A \oplus C.$$

eg. non split SES: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$

③ What is Tor?

"Tor" $= \mathrm{Tor}_1^{\mathbb{Z}}$ $\quad (\mathrm{Tor}_i^{\mathbb{Z}} = 0 \ \ \forall i \geq 2)$

For $\mathbb{Z}$-modules $A, B$:

- $A$ or $B$ is free $\Rightarrow$ $\mathrm{Tor}(A,B) = 0$
- If $A, B$ are finitely-generated, then
$$\mathrm{Tor}(A,B) = A_{tor} \otimes B_{tor}$$
where $A_{tor}$, $B_{tor}$ are the torsion subgroups of $A, B$ respectively.

## Künneth's Formula applied to CW cpxs

Let $X, Y$ be CW cpxs built with cells

$\{e^i_{X,\alpha}\}$ and $\{e^j_{Y,\beta}\}$, respectively.

Then $X \times Y$ has an induced cell complex structure with cells $\{e^i_{X,\alpha} \times e^j_{Y,\beta}\}$.

We get an identification of chain cpxs

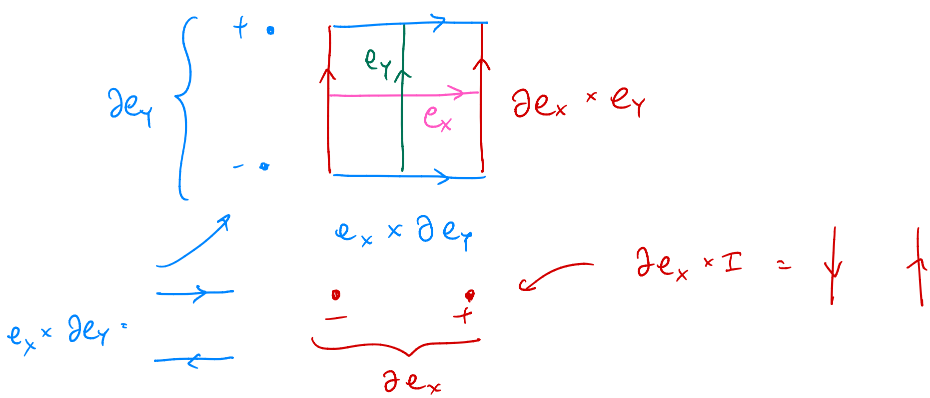
$$C^{CW}_\bullet(X \times Y) \cong C^{CW}_\bullet(X) \otimes C^{CW}_\bullet(Y)$$

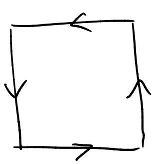
$$(e^i_{X,\alpha} \times e^j_{Y,\beta}) \longmapsto e^i_{X,\alpha} \otimes e^j_{Y,\beta}$$

by observing that $\partial(e^i_{X,\alpha} \times e^j_{Y,\beta})$

agrees with $\partial^{Tot}(e^i_{X,\alpha} \otimes e^j_{Y,\beta})$:

schematic/ example $(i,j=1)$:




$\partial(e_X \times e_Y) =$  $= \partial e_X \times e_Y + (-1)^1 e_X \times \partial e_Y.$

Applying the Künneth formula, we have

<u>thm</u> (Künneth Formula for CW cpxs):

$$H_n(X\times Y) \cong \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y) \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}\big(H_i(X), H_j(Y)\big).$$

<u>rmk</u> By cellular approximation, we also have the Künneth formula for general spaces, not just CW cpxs.

## Homology with Coefficients

Let $G$ = abelian group, $X$ = space

we can define the singular chain cpx over $G$ coeffs:

- $C_n(X;G) = C_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G$
- differential: $\partial \otimes 1$
- when augmenting the chain cpx for reduced homology, augment by $G \cong \mathbb{Z} \otimes_{\mathbb{Z}} G$.

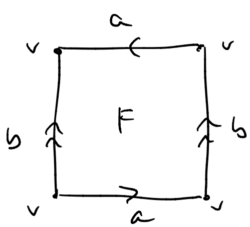
The resulting $H_n(X;G)$ are the <u>homology groups of $X$ with coefficients in $G$</u>.

<u>Rmk.</u> usually use $G = \mathbb{Z}, \mathbb{F}_2, \mathbb{Q}$.

$\mathbb{F}_2$ is particularly useful for nonorientable manifolds:

eg. $\mathbb{RP}^n$ over $\mathbb{F}_2$; HW?

eg. Klein bottle




Over $\mathbb{F}_2$:

$\partial_2(F) = 2a = 0$.

$\Rightarrow \forall i,\ \partial_i = 0$!

$$0 \to \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \oplus \mathbb{F}_2 \xrightarrow{0} \underline{\mathbb{F}_2} \to 0.$$

Homology over $\mathbb{F}_2$ looks like that of the torus!

whereas over $\mathbb{Z}$, $H_1(\text{Klein bottle}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_2 = 0$.

The Universal Coefficient Theorem tells us that $H_\bullet(X;G)$ is determined by $H_\bullet(X;\mathbb{Z})$.

**thm** (Universal Coefficient Theorem for Homology)
There is a split SES $\forall i$

$$0 \to H_i(X;\mathbb{Z}) \otimes G \longrightarrow H_i(X;G) \longrightarrow \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), G) \longrightarrow 0.$$

Pf. Let $\mathcal{A} = C_\bullet(X;\mathbb{Z})$ and

$\mathcal{B}$ = a free resolution of $G$ i.e. free chain cpx s.t. homology is $0 \to \underline{G} \to 0$.
(e.g. use a group presentation for $G$).

Apply Künneth formula to $\mathcal{A} \otimes \mathcal{B}$. ∎

HW: examples where you can use UCT.

F 2.13.26

Now consider more regular topological spaces

<u>defn</u>

A <u>manifold of dimension $n$</u>, ie an <u>$n$-manifold</u>, is a Hausdorff space $M$ where each point has an open nbhd homeo to $\mathbb{R}^n$

eg. line with two origins is not a manifold

- notice this definition doesn't allow for boundary; thats a manifold w/ $\partial$, where some points can have nbhds that look like $[0,\infty) \times \mathbb{R}^{n-1}$

- A compact manifold is called <u>closed</u> (as we have already discussed)

more alg top comment.

Alg-top interpretation of dimension of manifold:

let $x \in M$. Then the local homology group $H_i(M, M-\{x\}; \mathbb{Z})$ is nonzero only for $i=n$:

$$H_i(M, M-\{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n-\{x\}; \mathbb{Z})$$

by excising everything outside of a nbhd $\cong \mathbb{R}^n$ of $x$.

$$\cong \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z})$$

by LES of pair, and since $\mathbb{R}^n \simeq *$.

$$\cong \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z})$$

$\mathbb{R}^n - \{0\} \simeq S^{n-1}$.

<u>Recall</u> LES:

$$\to H_i(\mathbb{R}^n - \{x\}) \to H_i(\mathbb{R}^n) \to H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \to H_{i-1}(\mathbb{R}^n - \{x\}) \to H_{i-1}(\mathbb{R}^n) \to \dots$$

# orientations & homology

* usual notion of orientation from diff top
  → frames, etc.
  → de Rham cohomology

Algebraic-topological defn of orientation:

<u>defn</u> An orientation of $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is a choice of generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathbb{Z}$ as discussed previously

<u>rmk</u>. For any two points $x, y \in \mathbb{R}^n$, we can find a ball $B$ containing both pts.
So the canonical isomorphisms

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$$

allow us to propagate one choice of orientation to all points.
"local consistency"

<u>local orientations for manifolds</u>:

<u>defn</u> A local orientation of $M$ at a point $x$ is a choice of generator $\mu_x$ of the infinite cyclic group $H_n(M, M - \{x\})$.

<u>notation</u> Hatcher uses $H_n(X|A) := H_n(X, X-A)$ for shorter notation.
This is intuitive, since we're kind of looking at $X$ restricted to $A$
we view $H_n(X|A)$ as the "local homology of $X$ at $A$".

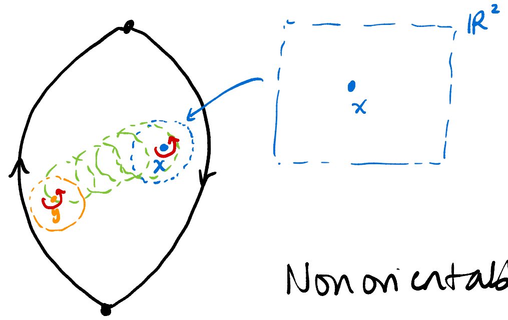
## global orientations for manifolds:

defn An orientation of $M^n$ is an function $x \mapsto \mu_x$ assigning to each point $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ satisfying the "local consistency" condition:

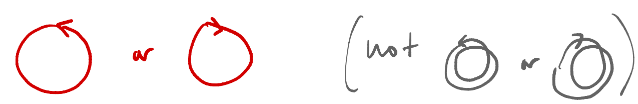
each $x \in M$ has a nbhd homeo to $\mathbb{R}^n \subset M$ ← metric space (viewing $\mathbb{R}^n \subset M$)
containing an open ball $B$ of finite radius containing $x$
s.t. all local orientations $\mu_y$ at points $y \in B$ are images
of one generator $\mu_B$ of $H_n(M|B) \cong H_n(\mathbb{R}^n|B)$
under the natural maps $H_n(M|B) \longrightarrow H_n(M|y)$.

If an orientation exists for $M$, then $M$ is orientable.

eg.




$H_2(\mathbb{R}^2|x)$
$\cong H_1(S^1)$




Nonorientable because if you propagate $\mu_x$ the other way the two orientations at $y$ disagree.

But we can extend the notion of orientability by using covers!
(need this to define fundamental class)

Recall.

defn An orientation of $M^n$ is an function $x \longmapsto \mu_x$ assigning to each point $x \in M$ a local orientation $\mu_x \in H_n(M|x)$

$$H_n(M|x) = H_n(M, M-\{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_{n-1}(\mathbb{R}^n - \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Fact Every manifold has an orientable 2-sheeted covering space $\tilde{M}$

eg. $S^2$ covers $\mathbb{RP}^2$, Torus covers Klein bottle

idea If $M$ is nonorientable, then there is some closed loop $\gamma$ you can travel along where the orientation is inconsistent.
We can "unwrap" $\gamma$ to get a 2sheeted cover.

Concrete construction of $\tilde{M}$:

As a set,

$$\tilde{M} = \{\mu_x \mid x \in M,\ \mu_x \text{ is a local orientation of } M \text{ at } x\}$$

- $\mu_x \mapsto x$ is clearly 2:1 surjection

- Topologize the set $\tilde{M}$ to make $\mu_x \mapsto x$ a covering space projection:

  given a ball of finite radius $B \subset \mathbb{R}^n \hookrightarrow M$
  and a generator $\mu_B \in H_n(M \mid B)$, let
  $$U(\mu_B) = \{\mu_x \in \tilde{M} \mid x \in B \text{ and } \mu_x \text{ agrees with } \mu_B\}$$

- $\mu_x \in \tilde{M}$ determines its own local orientation:
  $H_n(\tilde{M} \mid \mu_x) \cong H_n(U(\mu_B) \mid \mu_x) \cong H_n(B \mid x)$ (all canonical)
  and by construction these local orientations "glue together" (satisfy the local consistency condition)
  $\Rightarrow \tilde{M}$ is orientable.

<u>prop</u> Let $M^n$ be connected.

① $M$ is orientable <u>iff</u> $\tilde{M}$ has two components (two copies of $M$)

② In particular, $M$ is orientable if $\pi_1(M)$ has no subgroup of index 2.

pf.

① Since $M$ is connected, $\tilde{M}$ must be 1 or 2 cpnts

$\boxed{\Leftarrow}$ If $\tilde{M}$ is 2 cpnts, then $\tilde{M} \cong M \amalg M$, so $M \subset \tilde{M}$ is orientable.

$\boxed{\Rightarrow}$ If $M$ is orientable, then it has exactly 2 orientations

(since $M$ is connected) Pick an orientation at a point & propagate.

Each global orientation determines a cpnt of $\tilde{M}$.

② Contrapositive: $M$ nonorientable $\Rightarrow$ $\tilde{M}$ is connected $\Rightarrow$ $\pi_1(M)$ has a subgroup of index 2.

□