W 2.18.24

defn. Let R be a commutative ring with identity.
An R-orientation of M assigns to each $x \in M$ a generator of $H_n(M|x;R) \cong R$, subject to local consistency.

i.e. $u \in R$ s.t. $Ru = R$.
i.e. $u \in R^\times$ (unit in R)

We can form a very large cover M_R of M consisting of all $\alpha_x \in H_n(M|x;R)$ (not just generators μ_x!)

- Topologize M_R: open sets generated by ($B \subset \mathbb{R}^n \subset M$)

$$U(\alpha_B) = \{ \alpha_x \mid x \in B,\ \alpha_x \text{ agrees with } \alpha_B \}$$

defn A continuous map $\begin{aligned} M &\longrightarrow M_R \\ x &\longmapsto \alpha_x \end{aligned}$ is called a section of the covering space.

From this point of view, an R-orientation of M is a section of this covering space M_R whose value at each x is a unit of R.

Structure of M_R

From UCT, there is a canonical isom

$$H_n(M|x;R) \cong H_n(M|x) \otimes R.$$

Each $r \in R$ determines a subcovering space M_r consisting of the points $\pm \mu_x \otimes r \in H_n(M|x;R)$ as x varies over M.

- If r has order 2, then $r = -r \Rightarrow M_r$ is a copy of M.
- otherwise, $M_r \cong \tilde{M}$ (two sheeted cover)

Observe

① If M is $\mathbb{Z}$-orientable, then it is R-orientable

$M_r \cong$ either M or $\tilde{M} = M \amalg M$. In either case we have a section.

② A nonorientable mfld is R-orientable as long as R contains a unit of order 2.

③ Every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable.

__thm__ Let M be a closed connected n-mfld. Then

(a) If M is R-orientable, then the map
$$H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$$
is an isomorphism $\forall x \in M$.

(b) If M is not R-orientable, then the map
$$H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$$
is injective, with image $\{r \in R \mid 2r = 0\}$ $\forall x \in M$.

(c) $H_i(M;R) = 0 \quad \forall\, i > n$.

(pf. omitted)

In particular,

- $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ orientable} \\ 0 & M \text{ nonorientable} \end{cases}$

- $H_n(M;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

__defn__ An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a __fundamental class__ for M.

COHOMOLOGY (INTRO)

Cohomology is algebraically very closely related to homology.

homology: chain complex:

$$\mathscr{C} = \left(\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} \underline{C_0} \longrightarrow 0. \right)$$

cohomology: cochain cpx: Over $G \in \mathbb{Z}$-mod coefficients:

$$0 \longrightarrow \underline{\mathrm{Hom}_{\mathbb{Z}}(C_0, G)} \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_1, G) \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_2, G) \longrightarrow \cdots$$

$$0 \longrightarrow \underline{C_0^*} \xrightarrow{\delta} C_1^* \xrightarrow{\delta} C_2^* \longrightarrow \cdots$$

(everything dualized)

As a result of dualizing, the cohomology functor

$$\left\{ \begin{matrix} \text{Top spaces,} \\ \text{cts. maps} \end{matrix} \right\} \xrightarrow{H^*(-)} \text{graded } \mathbb{Z}\text{-mod}$$

is <u>contravariant</u>:

a map of spaces $f: X \to Y$ determines

a map $f^*: H^*(Y) \longrightarrow H^*(X)$ on cohomology.

Contravariance allows us to define the cup product, giving cohomology an algebra structure!

The cohomology groups $H^*(\mathcal{C})$ are determined by the homology groups (over $\mathbb{Z}$) via a universal coefficient theorem:

Thm 3.2 (Universal Coefficient Theorem for Cohomology)

If a chain cpx $\mathcal{C}$ of free abelian groups has homology groups $H_n(\mathcal{C})$ (ie over $\mathbb{Z}$), then the cohomology groups $H^n(\mathcal{C}; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by split exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \to \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$$

Treat Ext as black box just as we did with Tor.

<u>Useful Properties of $\mathrm{Ext}(H, G)$:</u>

- $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$
- $\mathrm{Ext}(H, G) = 0$ if H is free
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

Cohomology of spaces

X = space $\quad G$ = abelian group

$C^n(X;G) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), G)$ singular n-cochains with coefficients in G

$\varphi \in C^n(X;G)$ assigns to each $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$.

$$\delta = \partial^* : C^n(X;G) \longrightarrow C^{n+1}(X;G) \qquad \text{coboundary map}$$
$$\varphi \longmapsto \varphi \circ \partial$$

Explicitly,

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi\left(\sigma|_{[v_0, \dots, \hat{v_i}, \dots, v_{n+1}]}\right)$$

learn about a space by considering functions on the space...

- $\delta^2 = 0$ because $\partial^2 = 0$
- $\ker\delta$ = cocycles, $\mathrm{im}\,\delta$ = coboundaries
  - $\varphi \in \ker\delta$ if $\delta\varphi = \varphi\partial = 0$, ie $\varphi = 0$ on boundaries

<u>Rmk.</u> $0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \to \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$

- When $n=0$, we have $H^0(X;G) \cong \mathrm{Hom}(H_0(X);G)$

  As we saw in the low-dim'l example, $\varphi \in \ker\delta$ iff it is a constant fn. on path cpnts.

  So $H^0(X;G) =$ fns from path cpnts of X to G

$$\cong \prod_{\substack{\text{path}\\ \text{cpnts}\\ \text{of } X}} G$$

- When $n=1$, $\mathrm{Ext}(H_0(X), G) = 0$ since $\overbrace{H_0(X)}^{\text{homology}}$ is free.

  So we also have $H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$.

$\cong \mathrm{Hom}(\pi_1(X), G)$, since G is already abelian