M 2.23.26

With field coefficients   let F = field.

Recall $C_n(X;F) = C_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F$

- As F-module, has basis the singular n-simplicies of X.
- $\mathrm{Hom}_F(C_n(X;F), F) \overset{\text{as abelian grps}}{\cong} \mathrm{Hom}_{\mathbb{Z}}(C_n(X), F)$

  since for both you just specify the values on the basis: singular simplicies

- By a generalization of the UCT to modules over a PID (rather than $\mathbb{Z}$),

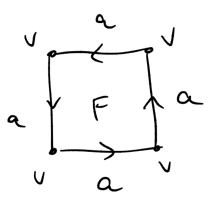
  we get

  $$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F), F)$$

  $\mathrm{Ext}_F(-,F)$ are all 0 since F is a field and all modules are free

Therefore: over a field, cohomology is exactly dual to homology.

eg.



Homology:

$$\begin{tikzcd}
 & 0 & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z} & \\
0 \arrow[r] & \langle F \rangle \arrow[r] & \langle a \rangle \arrow[r] & \langle v \rangle \arrow[r] & 0 \\
 & F \arrow[r, mapsto] & 4a & & \\
 & & a \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

Cohomology:

$$\begin{tikzcd}
 & \mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & \\
0 \arrow[r] & \langle v^* \rangle \arrow[r] & \langle a^* \rangle \arrow[r] & \langle F^* \rangle \arrow[r] & 0 \\
 & v^* \arrow[r, mapsto] & \underset{=0}{\delta v^* = v^* \partial} & &
\end{tikzcd}$$

$v^* \partial (a) = v^*(0) = 0$

$$a^* \longmapsto \delta a^* = a^* \partial = 4F^*$$

$a^* \partial (F) = a^*(4a) = 4$

$\Rightarrow a^* \partial = 4F^*$

ex. use "UCT" (relating homology and cohom) to deduce this relationship!

Compare over $\mathbb{F}_2$, $4 = 0$ so

$$\begin{tikzcd}
\text{homology:} & \mathbb{F} \arrow[r] \arrow[d, leftrightarrow, "\text{dual}"] & 0 \arrow[r] & \underline{\mathbb{F}} \arrow[d, leftrightarrow] \\
\text{cohomology:} & \mathbb{F} & 0 \arrow[l] & \underline{\mathbb{F}} \arrow[l]
\end{tikzcd}$$

We now think through all the constructions from homology and realize they all make sense or have analogous statements in cohomology.

Reduced Cohomology groups

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

dualize

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{\varepsilon^*} \mathrm{Hom}(C_0(X), G) \xrightarrow{\delta} \cdots$$

$$(f : \mathbb{Z} \to G) \longmapsto \left(f \circ \varepsilon : C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{f} G\right)$$

<u>note</u> $f \circ \varepsilon(\sigma) = f(1)$

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$$

$$0 \to \mathrm{Ext}(\underbrace{\tilde{H}_{-1}(X)}_{=0}, G) \to \tilde{H}^0(X; G) \longrightarrow \mathrm{Hom}(\tilde{H}_0(X), G) \longrightarrow 0$$

$$\Rightarrow \quad \tilde{H}^0(X; G) \cong \mathrm{Hom}(\tilde{H}_0(X), G)$$

Interpretation of $\tilde{H}^0(X;G)$:

from last time:

$\{\varphi : C_0(X) \to G \mid \text{constant on path cpnts}\}$

$$\tilde{H}^0(X;G) = \frac{\ker \delta_0}{\operatorname{im} \varepsilon^*}$$

$\varphi \in \operatorname{im} \varepsilon^*$ if $\varphi = f \circ \varepsilon$ for some f, i.e.

$\varphi(\sigma) = f(1) \ \forall \sigma.$ (see note above)

$\Longrightarrow \operatorname{im} \varepsilon^* = \{\varphi : C_0(X) \to G \mid \varphi \text{ is const on all } \sigma\}$

$$\Longrightarrow \tilde{H}^0(X;G) = \frac{\{\varphi : C_0(X) \to G \mid \text{constant on path cpnts}\}}{\{\varphi \text{ that are constant on all of } X\}}$$

(a sense of what it's like to dualize)

Relative groups, LES of a pair.

- Start with SES for pair (X,A)

This SES in a way defines $C_n(X,A)$

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$

Now apply $\text{Hom}(-,G)$.

- **Note** A priori, $(-)^* = \text{Hom}(-,G)$ is left-exact i.e.

If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact,

then $0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$ is exact

(but $B^* \xrightarrow{\alpha^*} A^*$ might not be surjective)

- But in our case,

$\text{Hom}(C_n(X,A), G)$

$$0 \longrightarrow \boxed{C^n(X,A;G)} \xrightarrow{j^*} C^n(X;G) \xrightarrow[\text{(restriction)}]{i^*} C^n(A;G) \longrightarrow 0$$

subgroup!

Is indeed exact:

Let $\varphi \in C^n(X;G)$. Then $i^*(\varphi) = \varphi|_A$.

Every $\psi : C_n(A) \to G$ can be extended by 0 to a function $\bar{\Psi} : C_n(X) \to G$. Then $i^*(\bar{\Psi}) = \psi$.

So i^* is indeed surjective.

- We can view $C^n(X,A;G)$ as functions

$$\{\text{singular } n \text{ simplicies in } X\} \longrightarrow G$$

that vanish on simplices in A.

- Relative coboundary maps $\delta : C^n(X,A;G) \longrightarrow C^{n+1}(X,A;G)$ are restrictions of the absolute $\delta : C^n(X;G) \to C^{n+1}(X;G)$ (to fns that vanish on simplicies in A).

So the same construction from SES ⟶ LES goes through:

$$\begin{tikzcd}
\cdots H^n(X,A;G) \arrow[r, "j^*"] & H^n(X;G) \arrow[r, "i^*"] & H^n(A;G) \arrow[dll, "\delta"] \\
H^{n+1}(X,A;G) \arrow[r] & \cdots &
\end{tikzcd}$$

To compute the connecting map δ:

extended

$$\begin{tikzcd}
 & & C^{n+1}(X,A;G) \arrow[d, "j^*"] & \bar{\varphi}\partial \arrow[d, mapsto] \\
\bar{\varphi} \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & C^n(X;G) \arrow[r] \arrow[d, "i^*"'] & C^{n+1}(X;G) & \bar{\varphi}\partial \\
\varphi & C^n(A;G) \arrow[uur, dashed] & &
\end{tikzcd}$$

$\varphi\partial = 0$

$$\delta[\varphi] = [\bar{\varphi}\partial].$$

- There is a relationship b/w the ~~cohomology~~ connecting maps!

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"'] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\mathrm{Hom}(H_n(A),G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(X,A),G)
\end{tikzcd}$$

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}),G) \to H^n(\mathcal{C};G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}),G) \to 0$$

Induced Homomorphisms

H^* is a contravariant functor.

$f: X \to Y$

↳ $f_\#: C_n(X) \longrightarrow C_n(Y)$

Hom(−, G)

↳ $f^\#: C^n(Y;G) \longrightarrow C^n(X;G)$

induces $f^*: H^n(Y;G) \longrightarrow H^n(X;G)$

- $\mathbb{1}^\# = \mathbb{1}$ so $\mathbb{1}^* = \mathbb{1}$.
- $(fg)^\# = g^\# f^\#$ so $(fg)^* = g^* f^*$

Homotopy Invariance

Recall we found a homotopy (prism operator) P s.t.

$$g_\# - f_\# = \partial P + P\partial.$$

Dualizing this relation we get

$$g^\# - f^\# = P^*\partial^* + \partial^* P^* = P^*\delta + \delta P^*$$

so P^* a chain htpy b/w $f^\#$ and $g^\#$.

$\Rightarrow$ they are chain htpic $\Rightarrow$ $f^* = g^*$ on cohom.

Excision

thm. Suppose $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$.

Then $i : (X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms

$$i^* : H^n(X,A;G) \longrightarrow H^n(X-Z, A-Z; G) \quad \forall n.$$

Recall in the pf we had $\rho i = \mathbb{1}$ and $\mathbb{1} - i\rho = \partial D + D\partial$.
Again dualize + use the 5-lemma.

Mayer-Vietoris Sequences

We still have the SES of duals

$$0 \to C^n(A+B;G) \longrightarrow C^n(A;G) \oplus C^n(B;G) \longrightarrow C^n(A\cap B;G) \longrightarrow 0$$

$$f \longmapsto (f|_A, f|_B)$$

$$(f,g) \longmapsto f|_{A\cap B} - g|_{A\cap B}$$

Recall: $C_n(A+B) \subset C_n(X)$, where inclusion is a chain htpy equivalence.

$$C^n(A+B;G) = \text{Hom}(C_n(A+B), G)$$

Taking duals, the restriction map

$C^n(X;G) \xrightarrow{i^*} C^n(A+B;G)$ is also a chain htpy equiv.

W 2.25.26.

Toward Cup product for Cohomology

Cross product in homology

$$H_i(X)\times H_j(Y) \longrightarrow H_{i+j}(X\times Y)$$

Induced by, in the CW cpx case,

$$\begin{array}{ccc} C_i(X)\times C_j(Y) & \longrightarrow & C_{i+j}(X\times Y) \\ (e^i_\alpha, e^j_\beta) & \longmapsto & e^i_\alpha \times e^j_\beta \end{array}$$

(Give $X\times Y$ the product cell structure)

But if we want to turn $H_*(X)$ into a ring, we need some natural map

$$H_{i+j}(X\times X) \longrightarrow H_{i+j}(X).$$

There isn't really a natural choice of map $X\times X \to X$ here.

On the other hand there is a very natural diagonal map $\Delta : X \to X\times X$

So we want a contravariant construction instead.

we can also define a cross product for CW cochains:

$$C^i(X)\times C^j(Y) \longrightarrow C^{i+j}(X\times Y)$$

$$(\varphi,\psi) \longmapsto \text{“}\varphi\times\psi\text{”}$$ where

$$(\varphi\times\psi)(e^i_\alpha\times e^j_\beta) = \varphi(e^i_\alpha)\cdot\psi(e^j_\beta)$$

multiplication in the <u>ring</u> $\mathbb{Z}$

We could check that this induces a product on cohomology.

$$H^i(X)\times H^j(Y)\xrightarrow{\times} H^{i+j}(X\times X)$$

The composition

\smile

$$\begin{tikzcd}
H^i(X)\times H^j(Y) \arrow[r, "\times"] \arrow[rr, bend left, "\smile"] & H^{i+j}(X\times X) \arrow[r, "\Delta^*"] & H^{i+j}(X)
\end{tikzcd}$$

turns out to be the <u>cup product</u>

Note that we have not checked many details. Most importantly, invariance under choice of cell structure!

Direct construction of cup product

Let R be a ring (eg. $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Q}$)

Cup product on cochains:

let $\varphi \in C^k(X;R)$

$\psi \in C^\ell(X;R)$

$\sigma : \Delta^{k+\ell} \to X$ generator of $C_{k+\ell}(X)$

Define

$$(\varphi \cup \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,v_{k+\ell}]}\right)$$

mult in R
(assoc, distributive)

Relate with differentials:

Lemma 3.6 $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$.

Pf. let $\sigma : \Delta^{k+\ell+1} \longrightarrow X$.

- $(\delta\varphi \cup \psi)(\sigma)$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{k+1}]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}\right)$$

last term:

✪ $(-1)^{k+1} \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}\right)$

- $(-1)^k(\varphi \smile \delta\psi)(\sigma)$

$$= \sum_{i=k}^{k+\ell+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]}\right)$$

w/ last term $(-1)^k \cdot \star$

These two terms cancel in the sum, and the remainder is $\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma)$. ∎

lemma 3.6 ⇒

① product of two cocycles is a cocycle:

If $\delta\varphi, \delta\psi = 0$ then $\delta(\varphi \smile \psi) = \cancelto{0}{\delta\varphi} \smile \psi \pm \varphi \smile \cancelto{0}{\delta\psi} = 0$

② product of cocycle and coboundary $= 0$ (either order)

- If $\delta\varphi = 0$ then $\varphi \smile \delta\psi = \pm\,\delta(\varphi \smile \psi)$
- If $\delta\psi = 0$ then $\delta\varphi \smile \psi = \delta(\varphi \smile \psi)$.

By ① & ②, $\smile$ descends to a cup product on cohomology:

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$$

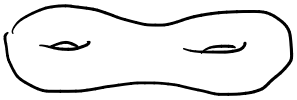
$$([\varphi],[\psi]) \longmapsto [\varphi \smile \psi]$$

$$(\varphi + \delta\varphi') \smile (\psi + \delta\psi') = \varphi \smile \psi + \text{coboundary}$$

This gives cohomology a graded ring structure, where the identity element is $1 \in H^0(X;R)$ rep'd by the cochain that sends all 0-simplicies to the identity $1 \in R$.

Recall $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$, $\sigma : \Delta^{k+\ell} \to X$

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

Example Σ_2 =  (can generalize to Σ_g)

① we know $H_*(\Sigma_2)$ is $H_0, H_1, H_2 \cong \mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}$, all free.

UCT $\Rightarrow$ $H^i(\Sigma_2) \cong \mathrm{Hom}(H_i(\Sigma_g), \mathbb{Z})$.

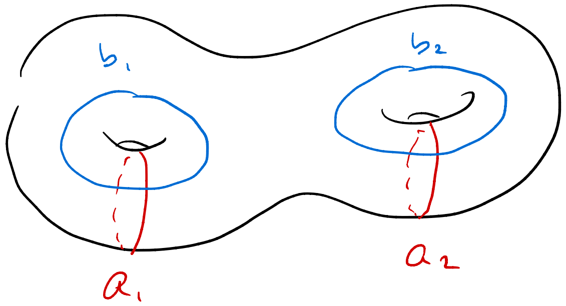
② Using Hatcher's notation, we'll write α_i as the dual to a_i, β_i dual to b_i, etc.

I personally prefer a_i^*; see HW. You can of course use whatever you want.

③ $H_0(\Sigma_2) \times H_0(\Sigma_2) \xrightarrow{\smile} H_0(\Sigma_2)$ is not very interesting; it's just multiplication in $\mathbb{Z}$.

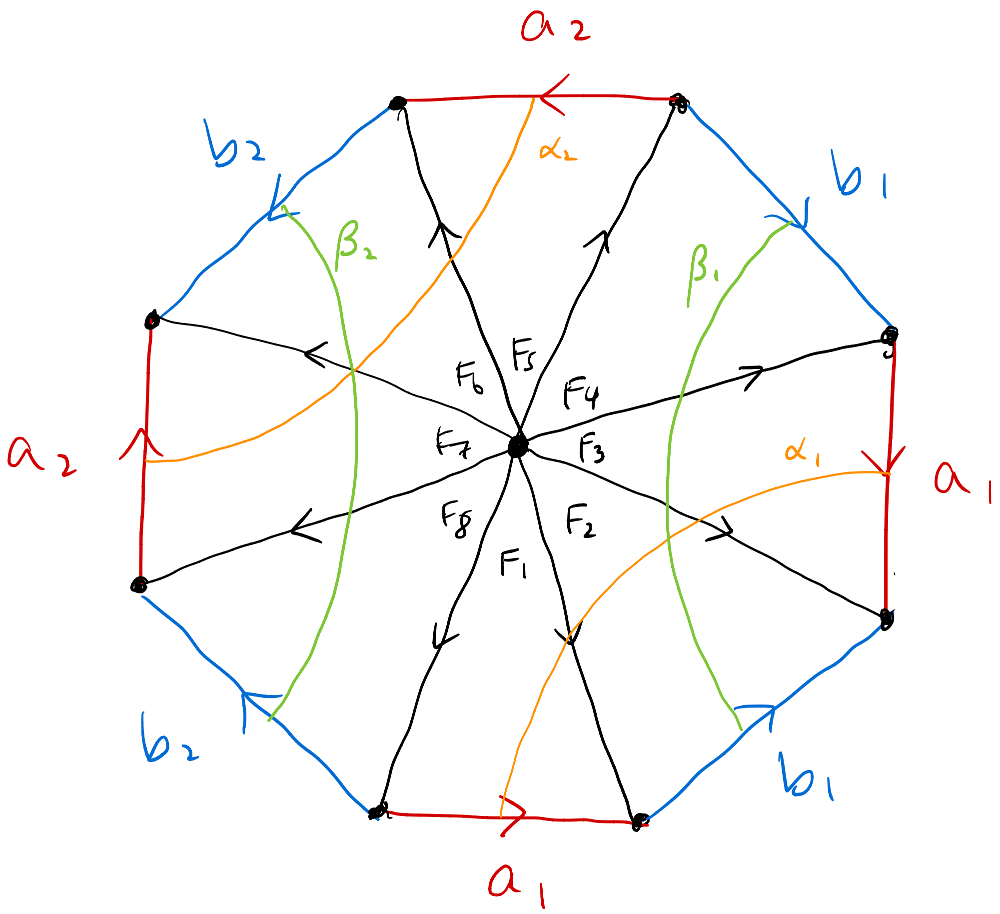
④ $H_1(\Sigma_2) \times H_1(\Sigma_2) \xrightarrow{\smile} H_2(\Sigma_2)$ is the only interesting cup product as $H_k = 0$ for $k > 2$.

Here we know $H_1(\Sigma_2)$ is generated by a_1, a_2, b_1, b_2



or better yet let's rep by the homologous cycles from the polygon description.

Simplicial structure on Σ_2 (that generalises to Σ_g)



(The black edges do not represent classes in homology, as we already know from CW homology.)

⑤ We now depict the duals $\alpha_i, \beta_i \in H^1(\Sigma_2; \mathbb{Z})$ by curves that transversely intersect a_i, b_i (respectively) exactly once.

α_i β_i drawn. – these are meant to rep. cohomology classes but by abuse of notation also denote the curves.

we can define simplicial cocycles φ_i, ψ_i representing the cohomology classes α_i, β_i resp.'ly by declaring

- The value of φ_i on a (simplicial) chain y = # times y intersects α_i
  - actually should give α_i a transverse orientation and count signed intersection, but note that the way its drawn, the intersecting curves all travel the same way across the curve α_i

- Similarly for ψ_i and β_i.

AND checking that these are actually cocycles!

The cocycle condition: $\delta\varphi_i = 0$ $\quad \delta\psi_i = 0$.

$$\delta\varphi([v_0, v_1, v_2]) \\ = \varphi([v_0, v_1]) - \varphi([v_1, v_2]) + \varphi([v_0, v_2]) = 0$$

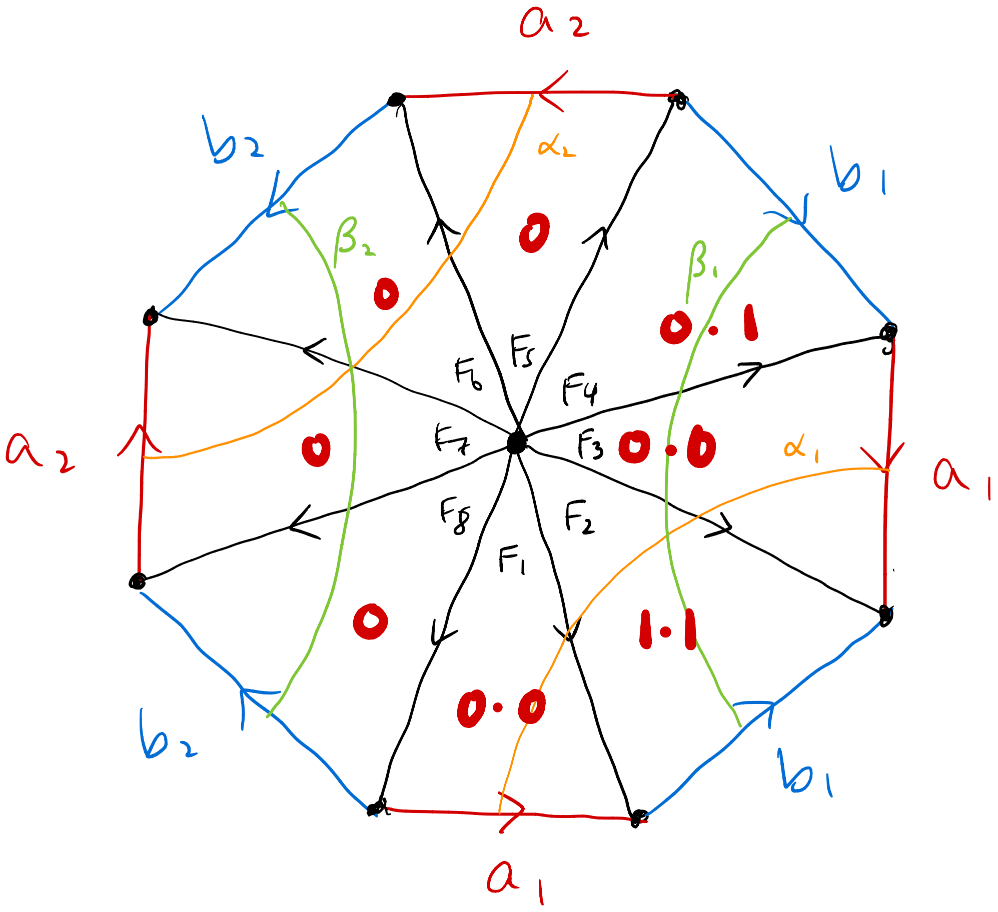
So the cocycle condition becomes

$$\varphi([v_0, v_1]) + \varphi([v_0, v_2]) = \varphi([v_1, v_2]). \qquad ✪$$

* check now that φ_i, ψ_i are all cocycles indeed. (we are lucky).
The α_i, β_i always $\cap$ a $[v_0, v_2]$, + then one else on ∂ of a Δ^2.

⑥ Now compute cup product at the cochain level.

* compute $\varphi_1 \smile \psi_1$ on each simplex. straight from definition of $\smile$.



So $\varphi_1 \smile \psi_1$ evaluates to 1 only on simplex F_2

What do we do w/ this info?

⑦ We need to find a representative of a generator of H_2.

Easy - we know from cellular homology that

$c = F_1 + F_2 - F_3 - F_4 + F_5 + F_6 - F_7 - F_8$

is a generator.

Then $(\varphi_1 \smile \psi_1)(c) = 1$. $(\varphi_1 \smile \psi_1 = c^*)$

So if γ is the dual to $[c]$

recall $H^2(\Sigma_2) \cong \mathrm{Hom}(H_2(\Sigma_2), \mathbb{Z})$

in our present example

then $\alpha_1 \smile \beta_1 = \gamma$.

Since $[c]$ is a generator of $H_2(\Sigma_2)$,

γ is a generator of $H^2(\Sigma_2)$

⑧ Let's compute the rest:

- $\psi_1 \smile \varphi_1 = ?$ only $=1$ on F_3.

  $\Rightarrow (\psi_1 \smile \varphi_1)(c) = -1$

  $\Rightarrow \beta_1 \smile \alpha_1 = -\gamma = -\alpha_1 \smile \beta_1$

rmk. This turns out to generalise - if R is commutative, then

if $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$ then

$$\beta \smile \alpha = (-1)^{k \cdot l} \alpha \smile \beta.$$

+ depends only on dimension (ie "degree")

↖ overloaded term here.

// Lect 22.

- $\alpha_2 \smile \beta_2 = \gamma$, $\beta_2 \smile \alpha_2 = -\gamma$ by symmetry
- $\alpha_1 \smile \alpha_1 = 0$ (compute together on picture)
- same w/ all the α_i, β_i; $\alpha_i \smile \alpha_j$, etc.

⑨ Conclude:

$$\alpha_i \smile \beta_j = \begin{cases} \gamma & i=j \\ 0 & o/w \end{cases} = -\beta_i \smile \alpha_j$$

$$\alpha_i \smile \alpha_j = 0, \quad \beta_i \smile \beta_j = 0.$$

Rmk. Observe that the cup product of two dual curves is only nonzero when they intersect!

↪ think about why this is true on a Δ^2

- they both need to leave the Δ^2 along $[v_0, v_2]$.

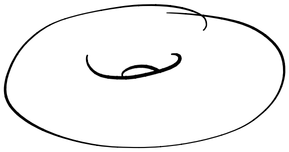
We can also say this for the same dual curve (see next example) by considering α with its pushoff α^+!

disjoint except at necessary transverse intersections

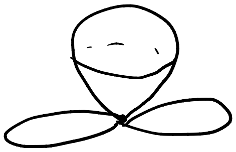
eg. In this example, α_i^+ is entirely disjoint from α_i

— same w/ the β_i.

⑩ significance: the cup product on cohomology distinguishes



from

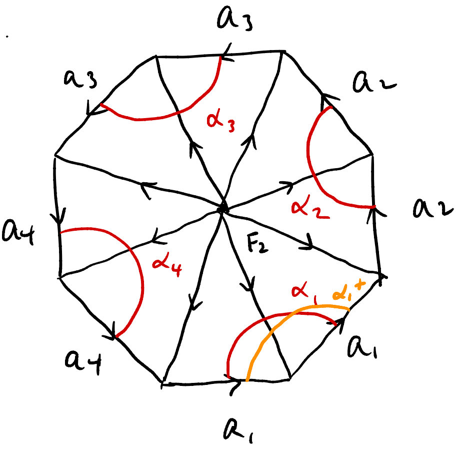


$\underbrace{S^1 \times S^1}_{\text{nontrivial cup product.}}$ $\underbrace{S^1 \vee S^2 \vee S^1}_{\text{trivial cup product.}}$

Rmk. Amazing coincidence for prev. example: could find the dual arcs that geometrically encode cocycle representatives

Sometimes we can't, at least over $\mathbb{Z}$ coefficients:

eg. 3.8.



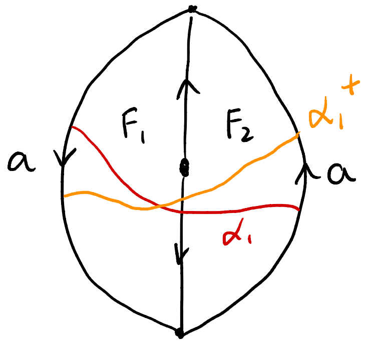
nonorientable closed surface of $\chi = 1 - 4 + 1 = -2$ (nonorientable genus 4)

Note that if we define φ_1 on cycles to be "# intersections with α_1" then φ_1 is not a cocycle, because $\delta\varphi_1(F_2) = 1 - 0 + 1 = 2$.

However, if we work over $\mathbb{F}_2$, then $2=0$ so φ_1 is a cocycle!

Let's compute the $\smile$ product for $\mathbb{RP}^2$ over $\mathbb{F}_2$.

eg.



Again define φ_1 on a cycle by the # intersections with α_1, mod 2.

- check that φ_1 is indeed a cocycle

$(\varphi_1 \smile \varphi_1)(F_1) = 0 \cdot 1$

$(\varphi_1 \smile \varphi_1)(F_2) = 1 \cdot 1$.

we are over field coeffs, so cohomology is dual to homology over $\mathbb{F}$ coeffs (recall).

The generator $c = F_1 + F_2$ (intuition from cellular homology) (ie $[c]$ generates $H_2(\mathbb{RP}^2; \mathbb{F}_2)$) has dual γ which generates $H^2(\mathbb{RP}^2; \mathbb{F}_2)$.

Since $(\varphi_1 \smile \varphi_1)(c) = 1$, $\boxed{\alpha_1 \smile \alpha_1 = \gamma}$.

In general:

- Σ_g = genus g orientable closed surface

$$\alpha_i \smile \beta_j = \begin{cases} \gamma & i = j \\ 0 & o/w \end{cases} = -\beta_i \smile \alpha_j$$

$$\alpha_i \smile \alpha_j = 0, \quad \beta_i \smile \beta_j = 0.$$

- N_k = nonorientable closed surface of $\chi = 2 - k$

$$\alpha_i \smile \alpha_i = \gamma, \quad \alpha_i \smile \alpha_j = 0 \text{ when } i \neq j$$

The cohomology ring

Since $\smile$ is an associative, distributive multiplication, we can associate a ring structure to $H^*(X)$!

It will be a graded ring: $H^*(X) = \bigoplus H^i(X)$ where the dimension is the grading.

multiplication is graded too as

$$gr(\alpha \smile \beta) = gr(\alpha) + gr(\beta)$$, indeed.

The constant function valued at $1 \in R$ represents the identity cohomology class.

Graded-commutative ring (when R commutative)!

$$\beta \smile \alpha = (-1)^{|\alpha||\beta|} \alpha \smile \beta$$

Cup product is natural:

Prop 3.10 $f : X \to Y$ map of top spaces

The induced maps

$$f^* : H^n(Y; R) \longrightarrow H^n(X; R)$$

satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Pf. Directly from definitions!

$$(f^\# \varphi \smile f^\# \psi)(\sigma)$$

$$= \underbrace{f^\# \varphi}_{\varphi f} \left(\sigma|_{[v_0, \ldots, v_k]}\right) \cdot \underbrace{f^\# \psi}_{\psi f} \left(\sigma|_{[v_{k+1}, \ldots, v_{k+l}]}\right)$$

$$= \varphi\left(f\sigma|_{[\ldots]}\right) \cdot \psi\left(f\sigma|_{[v \ldots]}\right)$$

$$= (\varphi \smile \psi)(f\sigma) = f^\#(\varphi \smile \psi)(\sigma).$$

F 2.26.24

eg. we computed $H^*(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^3)$

polynomials in α, where $\alpha^2 = \gamma$, recall.

thm. $H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[\alpha]$

$\uparrow$ $K(\mathbb{Z}/2\mathbb{Z}; 1)$. (Moore space; will discuss later)

$\uparrow$ group cohomology of $\mathbb{Z}/2\mathbb{Z}$

eg. $H^*(T^2) = \Lambda_{\mathbb{Z}}[\alpha, \beta]$ "Exterior algebra over $\mathbb{Z}$"

$\uparrow$ $\alpha \smile \alpha = 0$ $\beta \smile \beta = 0$

$\alpha \smile \beta = -\beta \smile \alpha$

In general $H^*(T^n) = \Lambda[\alpha_1, \dots, \alpha_n]$ ...

How do we compute this?

Künneth Formula for cohomology!

(Künneth Formula)

Cross product aka external cup product

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H^*(X \times Y) & \\
H^k(X) \arrow[ur, "p_1^*"] & & H^k(Y) \arrow[ul, "p_2^*"']
\end{tikzcd}$$

$$H^k(X;R) \times H^l(Y;R) \xrightarrow{\times} H^{k+l}(X \times Y; R)$$

$$(a,b) \longmapsto p_1^*(a) \smile p_2^*(b) =: a \times b$$

But note that since $\smile$ is distributive, the cross product is bilinear!

So by universal property of tensor products, there is a uniquely determined map (also called cross product)

$$H^k(X;R) \otimes H^l(Y;R) \xrightarrow{\times} H^{k+l}(X \times Y; R)$$

$$a \otimes b \longmapsto p_1^*(a) \smile p_2^*(b)$$

$$= a \times b$$

This is the map from the Künneth formula for cohomology!

thm 3.16 $H^*(X;R) \otimes_R H^k(Y;R) \longrightarrow H^*(X \times Y; R)$ is an isom of rings if X and Y are CW cpxs and $H^k(Y;R)$ is a f.g. free R-module $\forall k$.

(This statement is less general than the SES formulation of the Künneth Formula).

eg. $H^*(T^n;\mathbb{Z}) = H^*(\underbrace{S^1\times S^1\times\cdots\times S^1}_{n \text{ copies}};\mathbb{Z})$.

$$H^*(S^1) = \Lambda[\alpha_1]$$

$$H^*(S^1\times S^1) = \Lambda[\alpha_1]\otimes\Lambda[\alpha_2] \cong \Lambda[\alpha_1,\alpha_2]$$

note Tensor product of graded-comm rings:

$$(a\otimes b)(a'\otimes b') = (-1)^{|b||a'|}\, aa'\otimes bb'.$$

Moore Spaces (Aside)

defn. A Moore space $M(G,n)$ is a space
G = abelian gp, $n \geq 1$

$H_n(M(G,n)) = G$ and $\tilde{H}_i(M(G,n)) = 0 \ \forall\ i \neq n$,
and if $n \geq 2$, is simply connected.

eg.
- $M(\mathbb{Z},n)$ is easy to build
- $M(\mathbb{Z}/k\mathbb{Z}, n)$ via degree k map of sphere } ✪ see clarification next week.
- In general need presentation of G to construct

eg $\mathbb{RP}^2$ is an $M(\mathbb{Z}/2\mathbb{Z}, 1)$

Rmk If you want a space level construction to be interesting (eg. have interesting cohom ring), don't want (wedge of) Moore spaces because they basically give only the data of the homology groups.

Compare An Eilenberg-Maclane space $K(\pi, n)$ has $\pi_n(X) = \pi$.
otherwise $\pi_k(X) = 0$
These (in my opinion) are much harder to think about.

eg. $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$
b/c S^∞ is contractible.
$\hookrightarrow$ group cohomology

} we'll expand more on this next week.

Cap Product (toward Poincaré Duality)

defn let X be an arbitrary space, R a coefficient ring. The cap product is an R-bilinear pairing for all $k \geq l$

$$\frown : \underset{\sigma}{C_k(X;R)} \times \underset{\varphi}{C^l(X;R)} \longrightarrow C_{k-l}(X;R)$$

given by

$$\sigma \frown \varphi = \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \cdot \sigma|_{[v_l,\dots,v_k]}$$

- One can check via a calculation that

$$\partial(\sigma \frown \varphi) = (-1)^l\left(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi\right)$$

and use this to show that the cap product on chains induces a cap product on homology:

$$H_k(X;R) \times H^l(X;R) \xrightarrow{\frown} H_{k-l}(X;R)$$

thm 3.30 (Poincaré Duality)

If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $PD : H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $PD(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.

Remarks.

①

Fact The homology groups of a closed (recall, cpct) mfld are all finitely generated.

Thus using the UCT, we can also say the following in terms of coh homology:

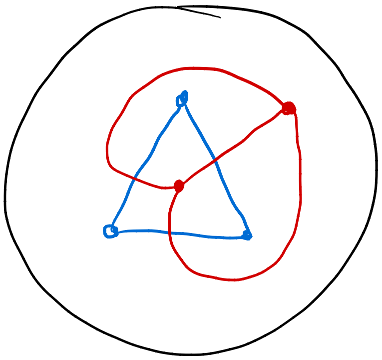
- M = orientable closed mfld.

Modulo torsion, $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ are isomorphic.

②

geometric idea: Dual Cell structures

eg. S^2 planar web gives a cell structure
dual web gives a dual cell structure, representing cochains



blue vertices ↔ red faces
blue edges ↔ red edges
blue faces ↔ red vertices

$$C_i \longleftrightarrow C^*_{n-i}$$

The interpretation that makes the most sense to me:
Morse / handle theory (only works for some manifolds)

end of class F 2.26.26