M 3.2.26 (clarifications from Friday, 2.27.26)

(I) Recall from 215A:

<u>prop. 4.1</u> A covering space projection $p:(\tilde{X},\tilde{x}) \to (X, x_0)$ induces isomorphisms $p_*: \pi_n(\tilde{X},\tilde{x}_0) \to \pi_n(X, x_0)$ $\forall n$.

<u>Consequence</u> $\forall n \geq 1$, $\pi_n(S^\infty) = 0$.

$\Rightarrow \pi_n(\mathbb{R}P^\infty) \cong 0 \quad \forall n \geq 2.$

And since $S^\infty \to \mathbb{R}P^\infty$ is 2-sheeted, we have $\pi_1(S^\infty) \trianglelefteq \pi_1(\mathbb{R}P^\infty)$ index 2 subgroup

$\Rightarrow \pi_1(\mathbb{R}P^\infty) \cong \mathbb{Z}/2\mathbb{Z}.$

$\Rightarrow \mathbb{R}P^\infty$ is an $K(\mathbb{Z}/2\mathbb{Z}, 1)$.

(as is anything hpy equiv to $\mathbb{R}P^\infty$)

(II) <u>thm</u> (Hurewicz theorem) If a space X is $(n-1)$-conn'ted, for $n \geq 2$, then $\tilde{H}_i(X) = 0$ $\forall i < n$ and $\pi_n(X) \cong H_n(X)$.

Pf. use LES for pairs for π_n and H_n.

<u>Cor 4.33</u> A map $f: X \to Y$ b/w simply-conn'ted CW cpxs is a hpy equiv if $f_*: H_n(X) \to H_n(Y)$ is an isom for each n.

* Consider proving (II).

next

(14) Claim Can build $M(\mathbb{Z}/k\mathbb{Z}, n)$ using degree k map of sphere. S^n

CW cpx w/ $\underbrace{e^0, e^n}_{S^n}, e^{n+1}$

$\varphi: \partial e^{n+1} \cong S^n \longrightarrow S^n$ by deg k map.

Then the CW homology chain cpx is

$$0 \to \underbrace{\mathbb{Z}}_{n+1} \xrightarrow{\cdot k} \underbrace{\mathbb{Z}}_{n} \longrightarrow \cdots \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

$\Rightarrow$ reduced homology is $\mathbb{Z}/k\mathbb{Z}$ @ grading n, 0 otherwise.

* lens space doesn't work b/c homology is $0 \to \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$.

Recall thm 3.30 (Poincaré Duality)

If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $PD: H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $PD(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.

* note M must have fundamental class.
eg. if M nonorientable, can take $R = \mathbb{F}_2$.

Important consequences of Poincaré duality: in my opinion...

① If M is orientable, then $\operatorname{rank} H_i(M;\mathbb{Z}) = \operatorname{rank} H^{n-i}(M;\mathbb{Z})$, which by the UCT is equal to $\operatorname{rank} H_{n-i}(M;\mathbb{Z})$.

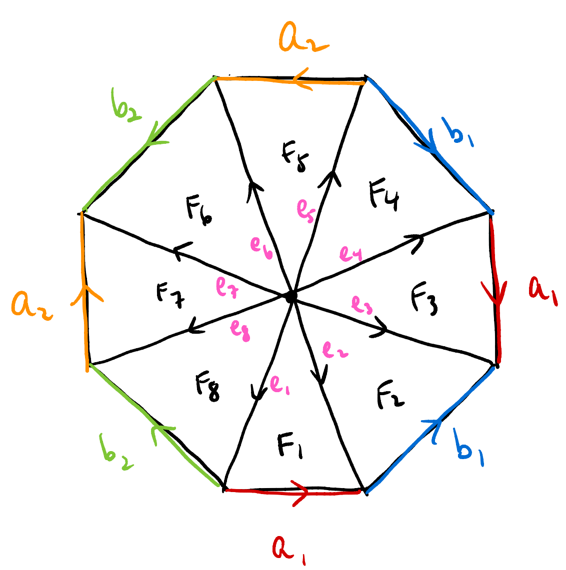
② A closed orientable odd dim'l mfld has $\chi(M) = 0$
Also true for nonorientable; use $\mathbb{Z}/2\mathbb{Z}$ coeffs.
(Corollary 3.37)

We will not prove Poincaré Duality in this class, but it's a fundamental fact from algebraic topology that is used all the time.

Instead let's look at an example:

Again, we actually do the example with simplicial (co)homology, and cap product has the same definition there.

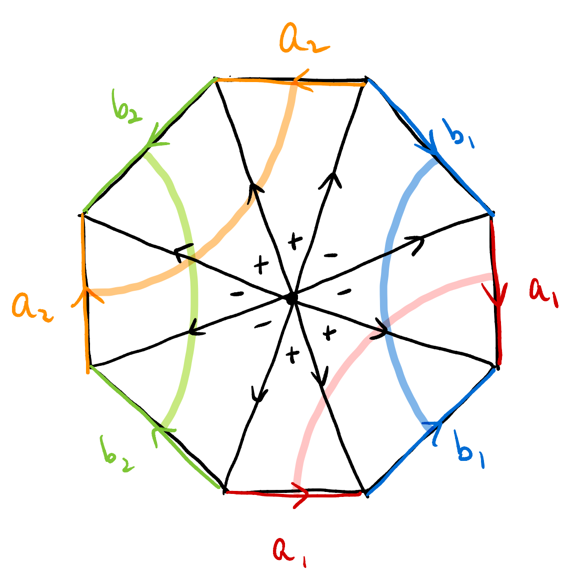
eg. Consider once again: genus 2 orientable surface.



The fundamental class $[M]$ is what we called "c" last time:

$$c = F_1 + F_2 - F_3 - F_4 + F_5 + F_6 - F_7 - F_8$$

(corresponding to the CCW orientation)



α_1 represented by cocycle φ_1

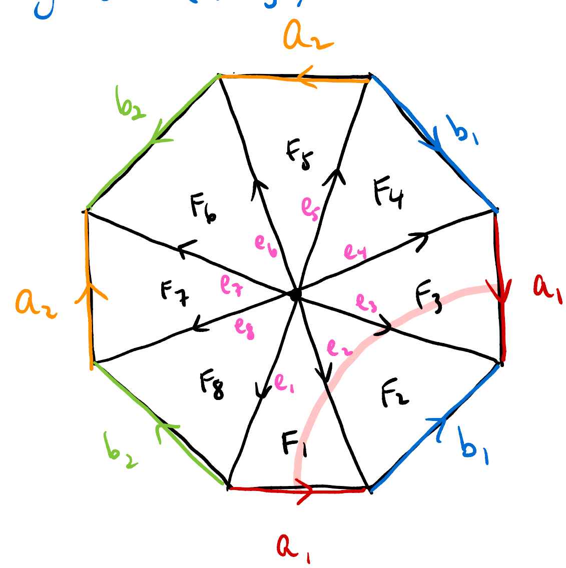
β_1 represented by cocycle ψ_1

α_2 represented by cocycle φ_2

β_2 represented by cocycle ψ_2

Consider $\frown : H_2(M) \times H^1(M) \longrightarrow H_1(M)$ to compute PDs:

$[M] \frown \alpha_1 = ?$ (Here I'm using $c \in [M]$)



$$\begin{cases} F_1 \frown \varphi_1 = \varphi_1(e_1)\cdot a_1 = 0. \\ F_2 \frown \varphi_1 = \varphi_1(e_2)\cdot b_1 = b_1 \\ F_3 \frown \varphi_1 = \varphi(e_4)\cdot a_1 = 0. \\ F_i \frown \varphi_1 = 0 \text{ for } i > 3 \end{cases}$$

$\Rightarrow c \frown \varphi_1 = b_1$

$\Rightarrow [M] \frown \alpha_1 = [b_1]$

Similarly, you can compute that

$$c \frown \varphi_i = b_i \qquad c \frown \psi_i = -a_i$$

So $PD(\alpha_i) = [b_i]$ and $PD(\beta_i) = -[a_i]$.

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Other forms of duality

① Poincaré-Lefschetz duality

Let M be an orientable compact n-mfld.

Then there are isomorphisms

$$H^k(M, \partial M; \mathbb{Z}) \cong H_{n-k}(M; \mathbb{Z})$$

$$H_k(M, \partial M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z})$$

again using cap product.

Poincaré duality is a special case.

② Alexander Duality

Let K be a compact, locally contractible, nonempty, proper subspace of S^n.

Then $\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z}) \quad \forall i.$

INTRO TO 4-MANIFOLDS   Reference: Gompf & Stipsicz

Let X be a compact, oriented (topological) 4-manifold.

$\Longrightarrow$ X has a fundamental class $[X] \in H_4(X, \partial X; \mathbb{Z})$

<u>defn</u>. The symmetric bilinear form

$$Q_X : H^2(X,\partial X;\mathbb{Z}) \times H^2(X,\partial X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined by $Q_X(a,b) = \langle a \smile b, [X] \rangle = a \cdot b \in \mathbb{Z}$
is called the <u>intersection form</u> of X.

evaluation of cohom class
at a hom class

$$Q_X(a,b) = \langle a \smile b, [X] \rangle = a \cdot b$$

(introducing notation)

a cohom. class in $H^4(X,\partial X;\mathbb{Z})$

a homology class in $H_4(X,\partial X;\mathbb{Z})$

<u>Notation</u>: Hatcher use $a, b, c \ldots$ for hom classes
$\alpha, \beta, \gamma \ldots$ for cohom classes.
Gompf - Stipsicz generally use the opposite convention. !!

Since by Poincaré-Lefschetz duality $H_2(X;\mathbb{Z}) \cong H^2(X, \partial X; \mathbb{Z})$, we can also think of Q_X as a pairing on $H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z})$.

Given $\alpha, \beta \in H_2(X;\mathbb{Z})$, if you can find surfaces

$$\Sigma_\alpha \xhookrightarrow{i_\alpha} X \qquad \text{and} \qquad \Sigma_\beta \xhookrightarrow{i_\beta} X$$

such that $i_{\alpha *}([\Sigma_\alpha]) = \alpha$ and $i_{\beta *}([\Sigma_\beta]) = \beta$, then

$$\alpha \cdot \beta = \Sigma_\alpha \cdot \Sigma_\beta ,$$

$\overset{2}{\text{codim}}\, \Sigma_\alpha + \overset{2}{\text{codim}}\, \Sigma_\beta = \overset{4}{\text{codim}}\, \Sigma_\alpha \cap \Sigma_\beta$

the algebraic intersection number of these two representing surfaces.

prop. Let X be a closed, oriented smooth 4-manifold. Then every element of $H_2(X;\mathbb{Z})$ can be represented by an embedded surface.

The pf relies on characteristic classes; see [Milnor-Stasheff] which relate cohomology groups of X with vector bundles over X.

Fact: Elements of $H^2(X;\mathbb{Z})$ are in 1-1 bijection with $U(1)$-bundles over X.

$\pi_1(X) = 0$
& simply cnted case easier to see geom'ly

For $\alpha \in H_2(X)$, let $a = PD(\alpha) \in H^2(X)$. Denote the corresponding $U(1)$-bundle by $L_\alpha \to X$.

Fact: The zero set of a generic section of the bundle $L_\alpha \to X$ will be a smooth surface representing α.

Now observe that since the target of

$$Q_X : \underset{\alpha}{\underset{\psi}{H_2(X;\mathbb{Z})}} \times \underset{\beta}{\underset{\psi}{H_2(X;\mathbb{Z})}} \longrightarrow \mathbb{Z}$$

is torsion-free, whenever a or b is a torsion element, by bilinearity we have $Q_X(\alpha,\beta) = 0$.

Hence Q_X descends to a bilinear pairing on homology mod torsion:

$$Q_X : \underbrace{\frac{H_2(X;\mathbb{Z})}{\text{tors}}}_{} \times \underbrace{\frac{H_2(X;\mathbb{Z})}{\text{tors}}}_{} \longrightarrow \mathbb{Z}$$

free $\Rightarrow$ can choose a basis!

By choosing a basis for Q_X, we can write Q_X as a matrix M.
Here are some important quantities related to M:
These are independent of the choice of basis.

- $\det Q_X := \det M$
- $\operatorname{rk}(Q_X) := \operatorname{rk}(H_2(X))$
- extend and diagonalize Q over $H_2(X) \otimes_{\mathbb{Z}} \mathbb{R}$.
    - $b_2^+ = \#\ +1$'s on the diagonal
    - $b_2^- = \#\ -1$'s on the diagonal
    - signature $\sigma(Q) = b_2^+ - b_2^-$
- parity: Q is <u>even</u> if $Q(\alpha,\alpha) \equiv 0 \pmod 2 \quad \forall\, \alpha \in H_2(X)$
  otherwise Q is <u>odd</u>.

prop. If X is a closed 4-manifold, then Q_X is unimodular, ie $\det(Q_X) = \pm 1$.

let $Q: A \times A \to \mathbb{Z}$ be a symm bilinear form.

Define $L_x : A \to \mathbb{Z}$ by $L_x(y) = x \cdot y = Q(x,y)$.

This defines a homomorphism $L: A \to A^* = \mathrm{Hom}(A, \mathbb{Z})$
$$x \longmapsto L_x.$$

Lemma. The form Q is unimodular iff L is an isom.

pf. Fix basis $a_1, \dots, a_n \in A$, dual basis $a_i^* \in A^*$

$a_i^*(a_j) = \delta_{ij} \in \mathbb{Z}$

Then $L(a_i) = \sum_j Q(a_j, a_i) a_j^*$.

The matrix for L is $[Q(a_i, a_j)]_{ij}$.

This is $\underline{\text{invertible over } \mathbb{Z}}$ iff L is an isom.

$\iff \det = \pm 1$. □

Idea of pf of prop

Consider the simplest setting algebraically: $\pi_1(X) = 0$ so that $H_1(X) = 0$ and we have no $\mathrm{Ext}(H_1(X), \mathbb{Z})$ to worry about.

Show that L is an isomorphism by showing that

$$L = H_2(X) \xrightarrow{PD} H^2(X) \xrightarrow{UCT} \mathrm{Hom}(H_2(X), \mathbb{Z})$$
$$\alpha \longmapsto \alpha^* \longmapsto (x \mapsto \alpha^*(x))$$

$PD(\alpha)(x)$

characterized by $[M] \frown \alpha^* = \alpha$

fund hom class. cohom hom class

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<u>defn.</u> Q is <u>positive</u> (resp. <u>negative</u>) <u>definite</u> if $rk(Q) = \sigma(Q)$ (resp. $rk(Q) = -\sigma(Q)$). Q is <u>indefinite</u> otherwise.

<u>thm.</u> If indefinite unimodular forms Q_1, Q_2 (defined on A_1, A_2 resp.) have the same rank, signature, and parity, then they are equivalent.

- If Q odd, then $Q \cong b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$
- If Q even, then $Q \cong \dfrac{\sigma(Q)}{8} E_8 \oplus \dfrac{rk(Q) - |\sigma(Q)|}{2} H$

This was all building up to the following mega theorem:

<u>thm</u> (Freedman 1982)

For every unimodular symmetric bilinear form Q, there exists a simply connected ($\pi_1(X)=0$), closed 4-mfld X such that $Q_X \cong Q$.

If Q is even, this manifold is unique (up to homeo)

If Q is odd, there are exactly two different homeomorphism types where Q is the intersection form, and at most one of these homeo types carries a smooth structure.

In particular, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms!

W 3.4.26

See KH notes for Morse theory idea.

Reference: Many; I originally learned from Milnor's orange book "Morse theory".

We then discussed relationship w/ handles (more on Friday)

and very briefly, Heegaard decompositions.

F 3.6.26.

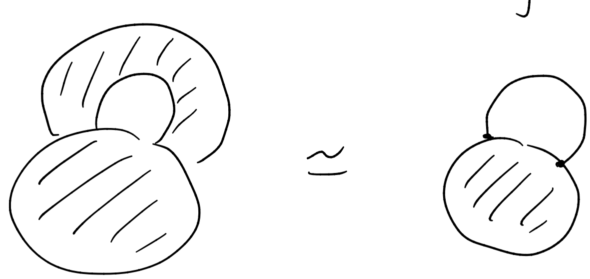
Reference: Gompf + Stipsicz Chp. 4.

defn. $0 \le k \le n$. An <u>n-dim'l k-handle h</u> is a copy of $D^k \times D^{n-k}$, attached to the boundary of an n-manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi : \partial D^k \times D^{n-k} \longrightarrow \partial X$.

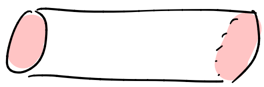
<u>Rmks</u>

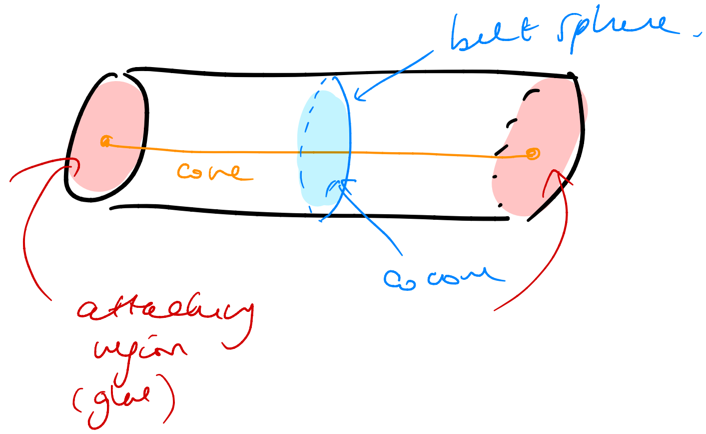
(1) Note $D^k \times D^{n-k}$ already has a smooth structure.
If X is a smooth mfld (w/ given smooth structure) and φ is smooth,
then
There is a canonical way to smooth corners
so that $X \cup_\varphi h$ has an induced smooth structure.

(2) Up to homotopy, attaching a k-handle is the same as attaching a k-cell.

eg. 

eg. 3-dim 1-handle

$h =$  $= \bullet\!\!-\!\!\!-\!\!\bullet \times D^2$



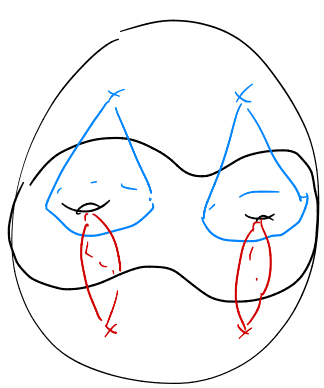
In general: n-dim'l k-handle $D^k \times D^{n-k}$

- attaching region $= \partial D^k \times D^{n-k}$
- core $= D^k \times pt$ (the k-cell core)
- co core $= pt \times D^{n-k}$
- belt sphere $= pt \times \partial D^{n-k}$.

Rmk Alternate names inspired by Morse theory:

- attaching sphere = "descending sphere"
- belt sphere = "ascending sphere"

cf. Heegaard decomp of 3mfld



We only care about the diffeomorphism type

= equiv class under diffeos

of $X \cup_\varphi h$, so we only care about the map

$$\varphi : \partial D^k \times D^{n-k} \longrightarrow \partial X$$

up to isotopy.

homotopy through diffeos.

dfn An isotopy b/w $\varphi, \varphi' : \partial D^k \times D^{n-k} \longrightarrow \partial X$ is a family of diffeos $\varphi_t : \partial D^k \times D^{n-k} \longrightarrow \partial X$ s.t. $\varphi_0 = \varphi$ and $\varphi' = \varphi_1$

Such an isotopy determines a diffeo

$$X \cup_\varphi h \approx X \cup_{\varphi'} h$$

up to ambient isotopy ...

Aside "Isotopy" and "ambient isotopy"

Consider a knot, ie embedding $K : S^1 \hookrightarrow S^3$

defn An ambient isotopy of K is a family of diffeos $S^3 \to S^3$

$$\Phi : I \times S^1 \hookrightarrow S^3 \qquad \varphi_i(t, s) = \Phi(i, s)$$

s.t. $\varphi_0 = id_{S^3}$.

Then $\varphi_1 \circ K : S^1 \hookrightarrow S^3$ is the isotoped knot.

We usually consider knots in S^3 up to isotopy, ie $K \sim \varphi_1 \circ K$.

By the isotopy extension thm, we can extend the isotopy of φ to an ambient isotopy

$$\Phi : I \times \partial X \to \partial X.$$

Q. (After class last time) Why do we want handles if we already have cells?

A. The thickening does add information!

Roughly: cells $\rightsquigarrow$ homotopical info

handles $\rightsquigarrow$ smooth structure.

e.g. $\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4$.

If we look at the 2 skeleton X^2, we see a sphere S^2: (the $\mathbb{CP}^1$)

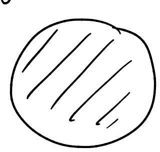
There is a unique way to glue $\partial e^2 \to e^0 = X^0$.

OTOH if we build using 4-dim'l handles, we get the smooth data of the tangent spaces too:

$$h^0 \cup h^2 = \nu(\mathbb{CP}^1) \subset \mathbb{CP}^2$$

nbhd of.

Specifically we want to glue

$h^2 =$  $\times D^2$ to $h^0 = D^4$

via a map

$$\varphi : \underbrace{S^1 \times D^2} \longrightarrow S^3$$

solid torus $\hookrightarrow S^3$.

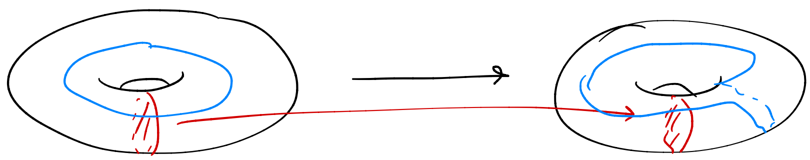
<u>Which map is it?</u> (what choice do we have?)

- First of all, $\varphi(S^1 \times *)$ must be unknotted so that ∂X^2 is an S^3

  – 3 mfld topology here

- But up to isotopy,

  homotopy through diffeos from identity.



Fact $X \cup_\varphi h$ is specified up to diffeo by 2 pieces of data:

① An embedding $\varphi_0 : S^{k-1} \longrightarrow \partial X$

* only CW attaching data!

ie a knot in ∂X

$\partial e^k = \partial$ of core.

with trivial normal bundle.

(to compare w/ later)

* new- smooth structure.

② a normal framing f of $\varphi_0(S^{k-1})$,

ie an identification of

$$\nu\, \varphi_0(S^{k-1}) \quad \text{with} \quad S^{k-1} \times \mathbb{R}^{n-k}$$

ie an ordered set of basis vectors at each pt.

$\Rightarrow$ $X \cup_\varphi h$ determined up to diffeo by the isotopy class of (φ_0, f).

Framings of S^{k-1} in ∂X^n w/ trivial normal bundle

define.

- Fix a framing f_0 to compare other framings to.

In many situations we can choose a canonical 0-framing f_0, eg. Seifert framing of knot in S^3

- then at any pt $p \in S^{k-1}$, we get an element $g \in \underbrace{GL(n-k)}_{\simeq O(n-k) \text{ by Gram Schmidt.}}$, $g(f_0) = f$.

<u>Note</u> we have a basepoint ($id(f_0) = f_0$).

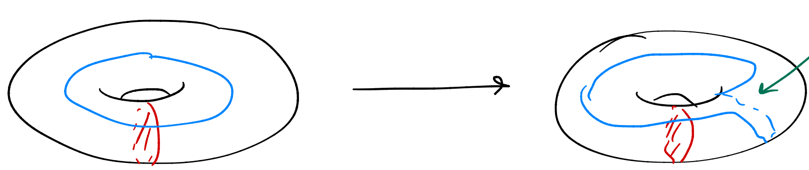
- $\Rightarrow$ we have a based map $S^{k-1} \longrightarrow O(n-k)$ and we consider it up to htpy

$\Rightarrow$ get an element of $\pi_{k-1}(O(n-k))$.

<u>eg.</u> $n=4$. $k=2$

$$\pi_{2-1}(O(4-2)) = \pi_1(O(2)) \cong \mathbb{Z}$$

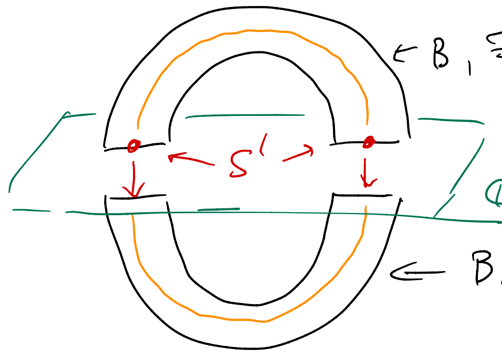
how many <u>Dehn</u> twists



Back to $\mathbb{CP}^2$ example $\quad X = \mathbb{CP}^2 \quad X^2 = 2\text{-skeleton} = \mathbb{CP}^1$.

$\nu(X^2) = \nu(\mathbb{CP}^1)$

View $\nu(\mathbb{CP}^1)$ as



$p \in B_0 \cap B_1 = [1 : w_1 : w_2]$ or $[z_1 : 1 : z_2]$

$\sim [1 : 1/z_1 : z_2/z_1]$.

Obs. $\quad w_1 = z_1^{-1}, \quad w_2 = z_1^{-1} z_2$

$\varphi(z_1, z_2) = (z_1^{-1}, z_1^{-1} z_2)$ is the attaching map.

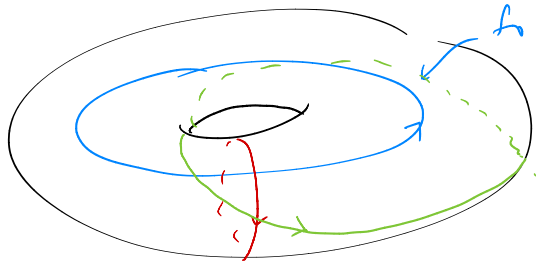
As we travel around the S^1 core of the attaching region, ($z_1 = e^{2\pi i t}$, $t \in [0,1]$)

the identification of the fibers

$$z_2 \longmapsto z_1^{-1} z_2 = e^{-2\pi i t} z_2$$

rotates around once.

$\hookrightarrow$ to get the sign, consider the sign of $\uparrow$ b/w the section and the 0-section.

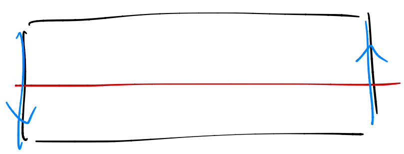


Recall fiber bundle / $\underbrace{\text{vector bundle}}$

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

locally $U \subset B$

$U \times \mathbb{R}^n$



Mob = I-bundle over S^1