MAT 215B Spring 2026
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Midterm Exam Solutions

1. Using the isomorphism between $H_*^{\Delta}(X)$ and $H_*(X)$, compute $H_*(X)$ for the triangular parachute obtained by identifying the three vertices of Δ^2. *Be careful. No points will be given for computing homology of the wrong space.*

   Solution. Let $\phi : \Delta^2 \to X$ be the quotient map. Let $v = \phi([v_0])$, $a = \phi([v_0, v_1])$, $b = \phi([v_1, v_2])$, $c = \phi([v_0, v_2])$, and $F = \phi([v_0, v_1, v_2])$.

   We compute the simplicial differentials:

   - ∂_0, $\partial_1 = 0$ (since there is only one vertex)
   - $\partial_2(F) = b - c + a$

   Hence the simplicial homology is

   - $H_0(X) \cong \mathbb{Z}$ since $\operatorname{im} \partial_0 = 0$
   - $H_1(X) \cong \mathbb{Z}\langle a, b, c\rangle / \langle b - c + a\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$
   - $H_2(X) \cong 0$ since F is not in the kernel.

2. Find a $\mathbb{Z}$-module A and maps f, g that make the following sequence exact:

$$0 \to \mathbb{Z} \xrightarrow{(\mathrm{id},0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0$$

   Solution.

   Since the image of $(\mathrm{id}, 0)$ is the first factor $\mathbb{Z} \oplus 0$, this sequence splits as the direct sum of the exact sequence

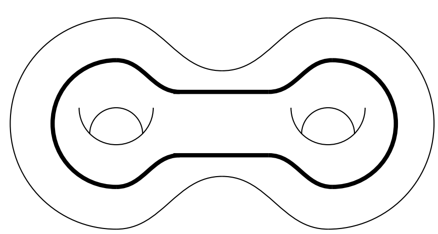
$$0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \oplus 0 \to 0$$

   and an exact sequence

$$0 \to 0 \oplus \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0.$$

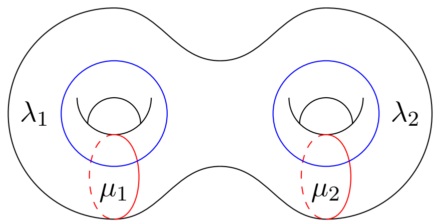
   Let $A = \mathbb{Z}$, let f be the map $\cdot n$, and let g be the quotient map $\mod n$. This is exact because f is clearly injective and g is clearly surjective, and $\ker g = \operatorname{im} f = n\mathbb{Z}$.

3. Determine the relative homology $H_*(X, A)$ for the pair (X, A) where X is the genus-2 surface shown below and $A =$ the thick circle shown.



Solution.

From CW homology we know that $H_1(X) \cong \mathbb{Z}^4$ is generated by the curves $\mu_1, \mu_2, \lambda_1, \lambda_2$ as shown below, where μ_i are red and λ_i are blue. Suppose μ_1, λ_1 are on the left, and μ_2, λ_2 are on the right.



Pick the counterclockwise orientation for A. Assume all curves are oriented counterclockwise as viewed in the drawing above.

We use the LES on homology of the pair (X, A), observing that $\widetilde{H}_0(A) = 0$ since A is path-connected, and that $H_i(A) = 0$ for $i \geq 2$ since A is 1-dimensional.

$$0 \longrightarrow H_2(X) = \mathbb{Z}\langle[\Sigma]\rangle \xrightarrow{j_*} H_2(X, A) \xrightarrow{\partial}$$

$$\longrightarrow H_1(A) = \mathbb{Z}\langle[A]\rangle \xrightarrow{i_*} H_1(X) = \mathbb{Z}\langle[\mu_1], [\mu_2], [\lambda_1], [\lambda_2]\rangle \xrightarrow{j_*} H_1(X, A) \longrightarrow 0$$

Observe that $i_* : H_1(A) \to H_1(X)$ takes the generator $[A] \mapsto [\lambda_1] + [\lambda_2]$ since the curve A is homologous to the curve $\lambda_1 \sqcup \lambda_2$ (since $-A$ and $\lambda_1 \sqcup \lambda_2$ bounds a pair of pants). Hence i_* is injective, so $\partial = 0$, so the LES above splits into two pieces.

The first piece is

$$0 \to \mathbb{Z} \xrightarrow{j_*} H_2(X, A) \xrightarrow{\partial} 0$$

which forces $H_2(X, A) \cong \mathbb{Z}$.

The second piece is

$$0 \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}^4 \xrightarrow{j_*} H_1(X, A) \to 0.$$

Hence $H_1(X, A) \cong \mathbb{Z}\langle[\mu_1], [\mu_2], [\lambda_1], [\lambda_2]\rangle / \langle[\mu_1] + [\mu_2]\rangle \cong \mathbb{Z}^3$.

Finally, since $H_0(A) \xrightarrow{i_*} H_0(X)$ is surjective, we have $H_0(X, A) = 0$.

4. Prove the 'invariance of dimension' theorem:

   Theorem. If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

   Hint: First relate $(U, U - \{x\})$ and $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$. Then note that $\mathbb{R}^m - \{x\} \simeq S^{m-1}$.

Solution.

Let $\phi : U \to V$ be a homeomorphism, pick a point $x \in U$, and let $y = \phi(x)$. Then ϕ induces a homeomorphism of pairs

$$(U, U - \{x\}) \cong (V, V - \{y\})$$

and hence an isomorphism on relative homologies.

Since U is open, there is an open ball of sufficiently small radius $B_x \subset U$ that contains x. There is homeomorphism of pairs $(B_x, B_x - \{x\}) \cong (\mathbb{R}^m, \mathbb{R}^m - \{x\})$. By excision of $U - B_x$, we have

$$H_*(U, U - \{x\}) \cong H_*(B_x, B_x - \{x\}) \cong H_*(\mathbb{R}^m, \mathbb{R}^m - \{x\}).$$

Running the same argument for $y \in V$, we have

$$H_*(V, V - \{y\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{y\}).$$

Since $\mathbb{R}^m \simeq \mathbb{R}^n \simeq *$, by LES of pairs we have

$$H_*(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong H_*(S^{m-1}) \qquad \text{and} \qquad H_*(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong H_*(S^{n-1})$$

Since spheres S^k have nontrivial homology in only dimensions 0 and k, for these two relative homologies to agree, we must have $m = n$.