MAT 150A Exam 2 Practice Solutions

Instructor: Melissa Zhang

1. Let $H = \{\pm 1, \pm i\} \leq \mathbb{C}^\times$.

   (a) Prove that H is normal in $\mathbb{C}^\times$.

   Since $\mathbb{C}^\times$ is an abelian group, every subgroup is normal.

   (b) Describe explicitly the cosets of H.

   For each positive radius $r \in \mathbb{R}^+$ and each angle $\theta \in [0, \pi/2)$, we have a coset $re^{i\theta}H$. (This coset is obtained visually by first rotating the four points of H CCW along the unit circle by θ and then scaling the circle by a factor of r.)

   (c) Identify the quotient group $\mathbb{C}^\times/H$. (*Hint:* If you're stuck, first play around with the map $\psi : S^1 \times S^1$ given by $e^{i\theta} \mapsto (e^{i\theta})^2$.)

   Consider the map $\varphi : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \mapsto z^4$. This is a homomorphism since $\varphi(xy) = (xy)^4 = x^4y^4 = \varphi(x)\varphi(y)$, for $x, y \in \mathbb{C}^\times$. This map is also surjective, since for any $z = re^{i\theta}$, the element $\sqrt[4]{r}e^{i\theta/4} \in \mathbb{C}^\times$ maps to z.

   Let $z = re^{i\theta} \in \ker\varphi$. Since $\varphi(z) = \varphi(r^4e^{4i\theta})$, it must be that $r = 1$ and 4θ is a multiple of 2π. Therefore $\ker\varphi = H$ precisely.

   By the First Isomorphism Theorem, $\mathbb{C}^\times/H \cong \mathbb{C}^\times$.

2. Let $\varphi : G \to G'$ be a surjective homomorphism between finite groups. Suppose $H \leq G$ and $H' \leq G'$ correspond to each other under the bijection in the Correspondence Theorem. Prove that $[G : H] = [G' : H']$.

   Fix the corresponding pair H and H'. Recall that H is a subgroup of G containing $N = \ker\varphi$, $H' = \varphi(H)$, and $H = \varphi^{-1}(H')$.

   To show $[G : H] = [G' : H']$, it suffices to give a bijective correspondence

$$\{\text{ left cosets of } H \text{ in } G\ \} \xrightarrow{f} \{\text{ left cosets of } H' \text{ in } G'\ \}. \tag{1}$$

   To a coset aH, define $f(aH)$ be $\varphi(aH)$. Since φ is a homomorphism, $\varphi(aH) = \varphi(a)\varphi(H) = \varphi(a)H'$, so $f(aH)$ is indeed a left coset of H'.

   We now show that f is a bijection. To see that f is injective, suppose two cosets aH and bH satisfy $f(aH) = f(bH)$. This means that $\varphi(a)H' = \varphi(b)H'$, so $\varphi(b)^{-1}\varphi(a) = \varphi(b^{-1}a) \in H'$. Hence $b^{-1}a \in H$, i.e. $aH = bH$. To see that f is surjective, let $a'H'$ be a left coset of H' in G'. Since φ is surjective, there exists $a \in \varphi^{-1}(a')$. Then $f(aH) = \varphi(aH) = \varphi(a)\varphi(H) = aH'$.

   Since f is a bijection, the cardinality of the left and right sides of (1) are the same, i.e. $[G : H] = [G' : H']$.

3. Let q be a 5-cycle in S_n, where $n \geq 6$.

(a) What is the cycle type of q^{17}? Since q is a 5-cycle, we may write

$$q = (a_1\ a_2\ a_3\ a_4\ a_5)$$

for some numbers $a_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq 5$. Since $17 \equiv 2 \mod 5$, the cycle notation for q^{17} is

$$q^{17} = (a_1\ a_3\ a_5\ a_2\ a_4),$$

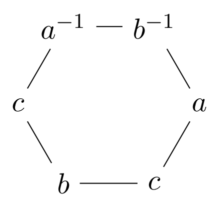
so q^{17} is also a 5-cycle.

(b) In terms of n, how many permutations are there such that $pqp^{-1} = q$? A conjugation pqp^{-1} of q is just a relabling of the numbers a_i above:

$$pqp^{-1} = (p(a_1)\ p(a_2)\ p(a_3)\ p(a_4)\ p(a_5)).$$

If we want $pqp^{-1} = q$, then the above 5-cycle must be taken to an equivalent 5-cycle, i.e. the cyclic order of the numbers a_i must be preserved. There are 5 such rotations. The value of p on the other $n - 5$ numbers does not affect the conjugation. Therefore there are $5(n - 5)!$ permutations p such that $pqp^{-1} = q$.

4. Prove that the conjugacy classes of a free group F_S (where S is the set of generators) are in bijection with the set of **closed words**, i.e. words that are written in a circle:



We can equivalently describe a closed word as an equivalence class of words under rotation, i.e. for a word $w = w_0 w_1 \cdots w_k \in F_S$, w is equivalent to any word w' of the form

$$w' = w_j w_{1+j} \cdots w_{k+j}, \tag{2}$$

where $1 \leq j \leq k$ and where the indices interpreted $\mod k+1$. (Of course, $w \sim w$ as well.) In other words,

$$w \sim w_2 w_3 \cdots w_k w_1 \sim w_3 w_4 \cdots w_1 w_2,$$

and so on.

Let $x \in F_S$. Then the conjugate xwx^{-1} is equivalent to $wx^{-1}x = w$, so $w \sim xwx^{-1}$.

Now suppose we are given w and w' in the same equivalence class, and write them as $w = w_0 w_1 \cdots w_k$ and $w' = w_j w_{1+j} \cdots w_{k+j}$ as above. Let $x = w_0 w_1 \cdots w_{j-1}$.

Conversely, given w and w' be in the same equivalence class. If $w' = w$, then we are done. So assume not, and write w' as a rotation of w as in (2). Consider the following conjugate of w':

$$\begin{aligned}
xw'x^{-1} &= (w_0 w_1 \cdots w_{j-1}) \cdot w \cdot (w_0 w_1 \cdots w_{j-1})^{-1} \\
&= (w_0 w_1 \cdots w_{j-1}) \cdot (w_j w_{1+j} \cdots w_{(k-j)+j}) \cdot (w_{(k+1-j)+j} \cdots w_{k+j}) \cdot (w_0 w_1 \cdots w_{j-1})^{-1} \\
&= w \cdot (w_{(k+1-j)+j} \cdots w_{k+j}) \cdot (w_0 w_1 \cdots w_{j-1})^{-1}
\end{aligned}$$

But $(k+1-j)+j = k+1 = 0 \mod k+1$, so the latter two factors are are inverses. Therefore $xw'x^{-1} = w$, and so w' and w are in the same conjugacy class of F_S.