MAT 150A Fall 2023
Instructor: Melissa Zhang
Exam 2

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ____________________________ Name (print): ____________________________

Name of left neighbor: ________________ Name of right neighbor: ________________

If you are next to the wall, then write "Wally" as your left or right neighbor. Write "Nemo" for your left/right neighbor if you don't have a left/right neighbor, respectively.

Question	Points	Score
Q1	25	
Q2	15	
Q3	20	
Total:	60	

- This is a **closed-book** exam. You may not use the textbook, cheat sheets, notes, or any other outside material. No calculators, computers, phones, or any other electronics are allowed.
- The last page of the exam packet is provided for scratchwork. **Do not detach** this sheet from your exam packet.
- You have **45 minutes** to complete this exam. If you are done early, you may leave after handing in your exam packet.
- Everyone must work on their own exam. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.
- **This is a proof-based course.** All statements must be justified and argued in the style of a mathematical proof. Failure to do so will result in the loss of correctness and/or style points.

1. Let C_{12} be generated by x and let C_6 be generated by y. Consider the surjective homomorphism $\varphi : C_{12} \to C_6$ determined by $x \mapsto y$. Explicitly write down the correspondence between subsets given by the Correspondence Theorem. *If you are claiming a group G has k subsets, you must explain (briefly) why you've found all of them.*

   Solution.

   First note that $\varphi(x^6) = y^6 = 1$, so $\{1, x^6\} \subset \ker\varphi$. Since $|\operatorname{im}\varphi| = 6$ and $|C_{12}| = 12$, we know $|\ker\varphi| = 2$. So $\ker\varphi = \{1, x^6\}$.

   The Correspondence Theorem tells us there is a bijection between the set of subgroups of C_{12} containing $\ker\varphi$ and the set of subgroups of C_6. In particular, for a subgroup $H \leq C_6$, $\varphi^{-1}(H)$ is the associated subgroup of C_{12} containing $\ker\varphi$.

   We first enumerate the subgroups of C_6. On a homework, we showed that if $H \leq C_6$, then H is cyclic. Therefore the subgroups of C_6 are

   (a) $\langle 1 \rangle$
   (b) $\langle y \rangle = \langle y^5 \rangle = C_6$
   (c) $\langle y^2 \rangle = \langle y^4 \rangle$
   (d) $\langle y^3 \rangle$.

   These correspond to the following subgroups of C_{12}, respectively:

   (a) $\varphi^{-1}(\langle 1 \rangle) = \langle 1, x^6 \rangle$
   (b) $\varphi^{-1}(\langle y \rangle) = \langle x \rangle = C_{12}$
   (c) $\varphi^{-1}(\langle y^2 \rangle) = \langle x^2 \rangle = \{1, x^2, x^4, x^6, x^8, x^{10}\}$
   (d) $\varphi^{-1}(\langle y^3 \rangle) = \langle x^3 \rangle = \{1, x^3, x^6, x^9\}$.

2. For each of the following, determine whether σ_1 and σ_2 are conjugate to each other in S_9. If they are conjugate, find a permutation $\tau \in S_9$ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

Note: 2a and 2b were removed from the exam.

(a) $\sigma_1 = (1\ 2)(3\ 4\ 5)$ and $\sigma_2 = (1\ 2\ 3)(4\ 5)$
Since σ_1 and σ_2 have the same cycle type, they are conjugate.
We can write

$$\sigma_1 = (1\ 2)(3\ 4\ 5)$$
$$\sigma_2 = (4\ 5)(1\ 2\ 3)$$

so that the cycles of the same length line up. We may choose τ to send $1 \mapsto 4, 2 \mapsto 5, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 3$. In cycle notation, $\tau = (1\ 4\ 2\ 5\ 3)$.
Other answers are possible.

(b) $\sigma_1 = (1\ 3)(2\ 4\ 6)$ and $\sigma_2 = (3\ 5) \circ (2\ 4)(5\ 6)$
In cycle notation, we compute that $\sigma_2 = (2\ 4)(3\ 5\ 6)$. This has the same cycle type as σ_1:

$$\sigma_1 = (1\ 3)(2\ 4\ 6)$$
$$\sigma_2 = (2\ 4)(3\ 5\ 6)$$

We may let τ send $1 \mapsto 2, 3 \mapsto 4, 2 \mapsto 3, 4 \mapsto 5, 6 \mapsto 6$. We can fill in the rest of the values of τ however we like. So one example of τ would be

$$\tau = (1\ 2\ 3\ 4\ 5).$$

(c) $\sigma_1 = (1\ 5)(7\ 2\ 4\ 3)$ and $\sigma_2 = \sigma_1^{2023}$

SOLUTION.
Since 2023 is $1 \mod 2$ and $3 \mod 4 = -1 \mod 4$, $\sigma_2 = \sigma_1^{2023} = (1\ 5)(7\ 3\ 4\ 2)$. This has the same cycle type as σ_1:

$$\sigma_1 = (1\ 5)(7\ 2\ 4\ 3)$$
$$\sigma_2 = (1\ 5)(7\ 3\ 4\ 2)$$

so one possible τ is $\tau = (2\ 3)$.

3. Let $G = (\mathbb{R}^2, +)$ and let $D \leq G$ denote the set of points on the diagonal:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ y = x\}.$$

(a) Briefly explain why $D \trianglelefteq G$.

SOLUTION.

Since G is an abelian group, any subgroup is normal.

(b) Use the First Isomorphism Theorem to identify the quotient group G/D with a familiar group.

SOLUTION.

We want to define a surjective homomorphism φ from G, such that $\ker \varphi = D$. So, let $\varphi : G \to \mathbb{R}^+$ be defined by $(x, y) \mapsto y - x$. Then for any $(x, x) \in D$, we have $\varphi(x, x) = x - x = 0$.

We check that φ is a surjective homomorphism. To see that φ is a homomorphism, we compute

$$\begin{aligned}
\varphi((x_1, y_1) + (x_2, y_2)) &= \varphi((x_1 + x_2, y_1 + y_2)) \\
&= y_1 + y_2 - x_1 - x_2 \\
&= (y_1 - x_2) - (y_2 - x_2) \\
&= \varphi(x_1, y_1) + \varphi(x_2, y_2).
\end{aligned}$$

For any $r \in \mathbb{R}$, φ sends $(0, r) \in \mathbb{R}^2$, so φ is surjective.

By the First Isomorphism Theorem, $G/\ker \varphi = G/D \cong \mathbb{R}^+$.

Scratchwork

Nothing on this page will be graded.