MAT 150A Final Exam Information

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Important Information

Sections C01 and C02: Monday, December 11, 2023, at 3:30 pm in Hoagland 168
Sections A01 and A02: Thursday, December 14, 2023 at 6:00 pm in Hoagland 168

- **You must attend the final exam with the section you are formally registered in.** There will be two versions of the exam, and I will only print out as many copies of each exam as the number of students enrolled in that section.
- Also, you are **not allowed** to discuss the exam with students outside of your section (e.g. on a Discord group)!

Topics Covered

The final exam is cumulative, and covers all the material introduced in Lectures 1–25 and Homeworks HW01–HW08. Material from the last three lectures (26–28) are also fair game if they only rely on concepts covered in Lectures 1–25; however, I would remind you of any relevant definitions from these last three lectures on the exam.

Remark. Keep in mind that "cumulative" means that I may ask you a question about a topic from later in the book which requires a technique from earlier in the book. For example, the following question would be fair:

> Let G be a group. Prove that the centralizer of s, $Z(s)$, is a subgroup of G.

This question requires you to know what it means for an element $g \in G$ to be in the centralizer of s, and also requires you to recall how to check whether a subset of G is a subgroup of G.

Below, we list the relevant book sections and the important concepts and terms introduced in that section. This list is not comprehensive, but may help represent what I personally feel is most important. You are not meant to do all the selected exercises (!); this just a list of problems you can try looking at if you are looking for more practice in material from a particular section.

Chapter 1

- Basic linear algebra: matrix multiplication and addition, what the columns of a matrix mean
- standard basis vectors, the vector space $\mathbb{R}^n$
- determinant of a 2×2 matrix, multiplicative property of determinants

- permutations (§1.5)
- inverse and transpose of a matrix

Chapter 2

- basics of group theory (§2.1, 2.2, 2.4)
  - groups, subgroups, laws of composition, cancellation law, etc.
  - abelian groups
  - order of an element
- important examples
  - $\mathbb{Z}^+ = (\mathbb{Z}, +)$; additive and multiplicative groups of real, complex numbers
  - general linear group $GL_n(\mathbb{F})$ for a field $\mathbb{F}$ (usually $\mathbb{F} = \mathbb{R}, \mathbb{C}$)
  - special linear group SL_n
  - circle group $S^1 \leq \mathbb{C}^\times$
  - symmetric group, alternating group, transpositions
  - cyclic groups (§2.4), cyclic group $\langle x \rangle$ generated by an element x
  - $\mathbb{Z}/n\mathbb{Z}$ and modular arithmetic (§2.9)
- subgroups of the additive group if integers (§2.3)
  - $n\mathbb{Z} \leq \mathbb{Z}^+$
  - (recall) the gcd of two integers (usually natural numbers), relatively prime integers, least common multiple, division algorithm
  - Theorem 2.3.3
- maps between groups (§2.5, 2.6)
  - homomorphisms, isomorphisms, automorphisms, endomorphisms
  - examples: identity, trivial, and inclusion homomorphisms
  - kernel, image, preimage / inverse image
  - left/right cosets, Proposition 2.5.8, Corollary 2.5.9
  - conjugation, conjugates; normal subgroups $N \lhd G$
  - commutator
  - (recall) equivalence relations and classes, partitions (§2.7); equivalence relation defined by a maps, e.g. relation between cosets of $\ker\varphi$ with elements of $\mathrm{im}\varphi$.
- more on cosets (§2.8, 2.10)
  - Corollary 2.8.3, Lemma 2.8.7
  - counting formula (2.8.8) and consequences (2.8.9 – 2.8.14)
  - correspondence theorem (Theorem 2.10.5)
- product groups (§2.11)

  - multiplication in product groups
  - characterizing product groups: Proposition 2.11.4
  - semi-direct products: see lecture notes; main examples are $O(2)$, dihedral groups, and $\text{Isom}(\mathbb{R}^2)$

- quotient groups (§2.12)
  - definition of quotient group, notation for quotient group elements
  - the canonical map $G \to \overline{G}$
  - multiplication in quotient group
  - First Isomorphism Theorem (Theorem 2.12.10) and how to use it

- a selection of exercises from Chapter 2: 2.2, 2.3, 2.6, 4.1, 4.2, 4.7, 4.9, 5.1, 5.2, 5.3, 5.4, 5.6, 6.2, 6.3, 6.6, 6.7, 6.8, 6.11, 7.6, 8.3, 8.4, 8.5, 8.7, 8.8, 10.2, 10.3, 10.4, 11.4, 11.5, 11.6, 11.7, 11.8, 12.2, 12.4, 12.5

Chapter 3

- vector spaces (§3.1, 3.3)
  - definition of vector spaces, subspaces
  - (recall) basis, standard basis, span of a collection of vectors, linearly independent/dependent

- fields (§3.2)
  - definition of field
  - examples: $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Q}[\sqrt{2}], \mathbb{Z}/p\mathbb{Z}$ where p is prime

Chapter 6: symmetries

- symmetries of plane figures (§6.1)
  - geometric vocabulary: rotation, reflection, glide reflection, translation

- isometries (§6.2, 6.3)
  - generators for $\text{Isom}(\mathbb{R}^2)$: 6.3.1
  - the orthogonal group $O(2) = S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$ where S^1 is the rotation subgroup
  - $\text{Isom}(\mathbb{R}^2) = T \rtimes O(2)$ where $T \cong \mathbb{R}^2$ is the translation subgroup
  - the homomorphism $\text{Isom}(\mathbb{R}^2) \to O(2)$
  - orientation, orientation-preserving/reversing isometries

- discrete subgroups of $\text{Isom}(\mathbb{R}^2)$ (§6.4, 6.5)
  - examples: cyclic and dihedral groups
  - statement of Theorem 6.4.1
  - discrete group G; translation group and point group of G
  - lattices, Theorem 6.5.5

- group actions (i.e. operations) (§6.7–6.11)
  - group actions
  - orbits, stabilizers
  - free ("faithful": 6.11.4), transitive actions
  - $G \curvearrowright G$ by left multiplication, by conjugation
  - $G \curvearrowright G/H$ for a (not necessarily normal) subgroup G
  - the counting formula in the context of group actions
  - permutation representation
- a selection of exercises from Chapter 6: 3.2, 4.1, 4.2, 4.3, 5.5, 5.6, 5.7, 7.3, 7.7, 9.1, 9.4, M.2

Chapter 7

- more on $G \curvearrowright G$ by conjugation (§7.2)
  - centralizer, conjugacy class
  - counting formula
  - class equation
  - conjugation in S_n (§7.5)
- group presentations (§7.9, 7.10)
  - free group generated by a set S
  - groups described by generators and relations
- a selection of exercises from Chapter 7: 1.1, 1.2, 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.8, 2.13, 2.14, 3.1, 3.2, 3.3, 3.4, 5.1, 5.3, 5.7, 5.9, 10.2, 10.3

Review Problems

Here are a few review problems. I will cover some of the solutions on Friday. Note that these review problems don't cover all the concepts we discussed in class this quarter.

The solutions are provided on the following pages.

1. Let G be any group, and fix an element $g \in G$. Show that there is a **unique** homomorphism $\mathbb{Z} \to G$ such that $1 \mapsto g$.

2. Let G be a group, and let $H \leq G$ and $N \trianglelefteq G$ be subgroups of G, where N is normal in G. Prove that $K = N \cap H$ is a normal subgroup of H.

3. Let $G = \{z \in \mathbb{C} \ : \ z^n = 1 \text{ for some } n \in \mathbb{N}\}$. Prove that for any fixed integer $k > 1$, the map $G \to G$ defined by $z \mapsto z^k$ is a surjective homomorphism, but is not an isomorphism.

4. Prove that D_{12} and S_4 are not isomorphic.

5. Consider the product group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Let $\Delta \leq G$ denote the diagonal subgroup

$$\Delta = \{(a, a, a) \in \mathbb{Z} \ : \ a \in \mathbb{Z}\}.$$

   Identify the quotient group G/Δ via an isomorphism to a familiar group.

6. An octahedron is a polyhedron with eight faces that are all equilateral triangles of the same size. Let G be the group of (orientation-preserving) rotational symmetries of the octahedron.

   (a) Show that $|G| = 24$.

   (b) What is the order of the stabilizer of a face f?

Solutions to Review Problems

1. Let G be any group, and fix an element $g \in G$. Show that there is a **unique** homomorphism $\mathbb{Z} \to G$ such that $1 \mapsto g$.

   Solution.

   Fix $g \in G$, and let $\varphi : \mathbb{Z} \to G$ be a homomorphism such that $\varphi(1) = g$. It suffices to show that the value of $\varphi(n)$ for all $n \in \mathbb{Z}$ is determined by $\varphi(1)$. Indeed, since φ is a homomorphism and 1 generates $\mathbb{Z}$,

   - $\varphi(0) = 1_G$, the identity element of G;
   - for $n > 0$, $\varphi(n) = \varphi(\underbrace{1 + 1 + ... + 1}_{n}) = \varphi(1)^n = g^n$;
   - $\varphi(-1) = \varphi(1)^{-1} = g^{-1}$;
   - and therefore, for $n < 0$, $\varphi(n) = \varphi(-1)^{-n} = g^n$.

   The value of φ on all inputs $n \in \mathbb{Z}$ is determined, so φ (exists and) is unique.

2. Let G be a group, and let $H \leq G$ and $N \trianglelefteq G$ be subgroups of G, where N is normal in G. Prove that $K = N \cap H$ is a normal subgroup of H.

   Solution.

   To show $K \trianglelefteq H$, let $k \in K$ and $h \in H$; we wish to show that $hkh^{-1} \in K$. Since $h, k, h^{-1} \in H$ and H is a subgroup, we immediately have $hkh^{-1} \in H$ by closure of H. Since $h \in G$ and $k \in N$, $hkh^{-1} \in N$ since $N \trianglelefteq G$. Therefore $hkh^{-1} \in N \cap H = K$. As k and h were arbitrary, this shows that $K \trianglelefteq H$.

3. Let $G = \{z \in \mathbb{C} \ : \ z^n = 1 \text{ for some } n \in \mathbb{N}\}$. Prove that for any fixed integer $k > 1$, the map $G \to G$ defined by $z \mapsto z^k$ is a surjective homomorphism, but is not an isomorphism.

   Solution.

   Note that if $z^n = 1$, then $|z| = 1$, and $z = e^{2\pi i m/n}$ for some $m \in \{0, 1, \dots, n-1\}$.

   Fix k. We first show that the map $\varphi : G \to G$ defined by $\varphi(z) = z^k$ is a homomorphism. Let $z, w \in G$. Since complex multiplication is commutative,

   $$\varphi(zw) = (zw)^k = z^k w^k = \varphi(z)\varphi(w),$$

   so φ is indeed a homomorphism.

   We next show that φ is surjective. Pick any $w \in G$; we need to find $z \in G$ such that $z^k = w$. Since $w \in G$, there is some n such that $w^n = 1$. Therefore

   $$w = e^{\frac{2\pi i m}{n}}$$

   for some $m \in \{0, 1, \dots, n-1\}$. Consider the complex number

   $$z = e^{\frac{2\pi i m}{nk}}.$$

Since $z^{nk} = 1$ and $z^k = w$, $z \in G$ and $\varphi(z) = w$. As w was arbitrary, we have shown that φ is surjective.

Finally, to show that φ is not an isomorphism, it suffices to show that it is not injective. As a counterexample, consider $z_1 = e^{\frac{2\pi i}{k}}$ and $z_2 = z_1^2$. We have $z_1^k = z_2^k = 1$, which shows that both numbers are in G, and also that φ of both is 1. So φ is not injective.

4. Prove that D_{12} and S_4 are not isomorphic.

Solution.

These are both groups of order 24, but we will show that only D_{12} contains an element of order 12. Since isomorphism preserve the order of elements, this will prove that $D_{12} \not\cong S_4$.

Consider the usual presentation for D_{12}:

$$D_{12} = \langle \rho, \tau \mid \rho^{12} = \tau^2 = (\rho\tau)^2 = 1 \rangle.$$

Here, ρ is a rotation of order 12, corresponding to a rotation of a 12-gon by $\pi/6$.

On the other hand, any permutation in S_4 must have one of the following cycle types: (1,1,1,1), (2,1,1), (2,2), (3,1,), (4). In each of these cases, the order of the permutation is at most 4 (maximized by the cycle type (4)). Therefore S_4 does not contain an element of order 12.

5. Consider the product group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Let $\Delta \leq G$ denote the diagonal subgroup

$$\Delta = \{(a, a, a) \in \mathbb{Z} \ : \ a \in \mathbb{Z}\}.$$

Identify the quotient group G/Δ via an isomorphism to a familiar group.

Solution.

Define a map $\varphi : G \to \mathbb{Z} \times \mathbb{Z}$ by $(a, b, c) \mapsto (a - b, a - c)$.

Observe that φ is indeed a homomorphism:

$$\begin{aligned}
\varphi((a, b, c) + (x, y, z)) &= \varphi((a + x, b + y, c + z)) \\
&= (a + x - (b + y), a + x - (c + z)) \\
&= ((a - b) + (x - y), (a - c) + (x - z)) \\
&= \varphi(a, b, c) + \varphi(x, y, z).
\end{aligned}$$

To see that φ is surjective, let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Then $(0, -m, -n) \mapsto (m, n)$.

The kernel of φ consists of triples (a, b, c) where $a - b = a - c = 0$, i.e. $a = b = c$. Therefore $\ker \varphi = \Delta$.

By the First Isomorphism Theorem, $G/\Delta \cong \mathbb{Z} \times \mathbb{Z}$.

6. An octahedron is a polyhedron with eight faces that are all equilateral triangles of the same size. Let G be the group of (orientation-preserving) rotational symmetries of the octahedron.

   (a) Show that $|G| = 24$.

   (b) What is the order of the stabilizer of a face f?

Solution.

(a) Let V be the set of vertices; then $|V| = 6$. Fix $v \in V$, and let $-v$ denote the vertex antipodal to v. The stabilizer of a vertex v is the set of rotations by $\pi/2$ about the line that goes through v and $-v$, so $|G_v| = 4$. The orbit of v is all of V, so $|\mathcal{O}_v| = 6$. By the Orbit-Stabilizer theorem, $|G| = 4 \cdot 6 = 24$.

(b) Since we are only considering rotational symmetries, the only motions that fix f are rotations by $2\pi/3$ around the line normal to f. Therefore $|G_f| = 3$.