

MAT 150A HW04

[add your name here]

Due Tuesday, 10/31/23 at 11:59 pm on Gradescope

Reminder. Your homework submission **must be typed** (TeX'ed) up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- Detexify:
<https://detexify.kirelabs.org/classify.html>

Covered in this HW This homework is meant to be completed with knowledge of material from Lectures 1-12, with the purpose of preparing students to better understand upcoming material.

Grading Some of the (parts of) problems will be graded in detail out of several points, and necessary feedback will be given. The rest will be graded out of 2 points.

Exercise 1

The purpose of this exercise is to strengthen your understanding of the term “**well-defined**” in mathematics.

Sometimes mathematicians ask whether a function is well defined. What they mean is this: “Does the rule you propose really assign to each element of the domain one and only one value in the codomain?”

- *The Art of Proof*, by Matthias Beck and Ross Geoghegan.

- (a) Prove that the following assignment is **not** a well-defined function between sets:

$$\begin{aligned}\varphi : \mathbb{Z}/10\mathbb{Z} &\rightarrow \mathbb{Z}/7\mathbb{Z} \\ \bar{k} &\mapsto \bar{k}.\end{aligned}$$

(Recall that \bar{k} denotes the equivalence class of k in $\mathbb{Z}/n\mathbb{Z}$.)

- (b) Prove that the following assignment **is** a well-defined function between sets:

$$\begin{aligned}\varphi : \mathbb{Z}/10\mathbb{Z} &\rightarrow \mathbb{Z}/5\mathbb{Z} \\ \bar{k} &\mapsto \bar{k}.\end{aligned}$$

SOLUTION.

Exercise 2

The purpose of this exercise is to strengthen your understanding of the term “**factors through**” in mathematics.

Definition. Let $\varphi : G \rightarrow G'$ be a group homomorphism. If there exist

- a group H
- a homomorphism $\alpha : G \rightarrow H$
- a homomorphism $\beta : H \rightarrow G'$

such that $\varphi = \beta \circ \alpha$, then we say that φ **factors through** H .

We usually capture this information in the form of a **commutative diagram**:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \alpha \downarrow & \nearrow \beta & \\ H & & \end{array}$$

Here, the descriptor “commutative” indicates that traveling along the horizontal arrow φ is the same as traveling the length-2 path $\beta \circ \alpha$.

- Prove that the homomorphism $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, $k \mapsto \bar{k}$, factors through $\mathbb{Z}/9\mathbb{Z}$.
- Prove that for all $\ell \in \mathbb{N}$, φ factors through $\mathbb{Z}/3^\ell\mathbb{Z}$.

SOLUTION.