Lecture 01

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MAT 150A

Welcome to MAT 150A

Please take a slip of paper from the front of the room.

- Write your full name on the top left corner.
- Write down any nicknames, pronouns, etc. that you'd like me to use when speaking with you.

At the end of lecture, place your slip in the pile corresponding to the first letter of your surname.

Syllabus

All class materials are accessible from the **class website:**

`https://www.melissa-zhang.com/Teaching/FA2023/MAT150A.html`

- syllabus
- class calendar
- homework
- lecture slides

What is modern algebra?

Algebra

Symbols represent quantities, objects, relations, etc.

- Translate specific problem into abstract algebraic problem.
- Solve problem in generality, i.e. abstractly, i.e. algebraically.
- Translate solution back to specific problem.

Example

Tamara has 35 coins in nickels and quarters. In all, she has $4.15. How many of each kind of coin does she have? [a]

[a] Source: https://www.chilimath.com/lessons/algebra-word-problems/coin-word-problems/

What is modern algebra?

Different subfields of algebra are used to solve different parts of problems.

Linear Algebra

All relationships between variables are *linear*, as opposed to quadratic, exponential, non-algebraic, etc.

- We can simplify nonlinear problems using linear approximations.
- *Matrix groups* are used to *represent* more complicated abstract groups.

What is modern algebra?

Modern or _Abstract_ Algebra

This is a more general term referring to all algebra beyond, say, solving single-variable equations.

- algebraic structures: groups, rings, fields, lattices, representations, ...
- relationships between them: homomorphisms, isomorphisms, sub- and quotient objects, products, etc.

MAT 150A focuses on **groups**, the most foundational algebraic structure among those listed.

Definition

[a] A *group* is a set G together with a law of composition that has the following properties:

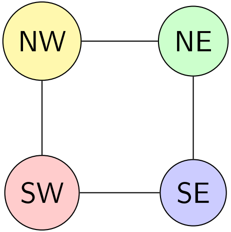
- is associative: $(a \quad b) \quad c = a \quad (b \quad c)$ for all $a, b, c \in G$.
- G contains an identity element e such that $e \quad a = a$ and $a \quad e = a$ for all $a \in G$.
- Every element $a \in G$ has an inverse, i.e. an element b such that $a \quad b = e$ and $b \quad a = e$.

[a] Page 40 in book

Note that commutativity of is not required.

Groups as sets of *symmetries*

What are the symmetries of a square?



Definition

The *dihedral group* D_{2n} is the group of symmetries of a regular n-gon.

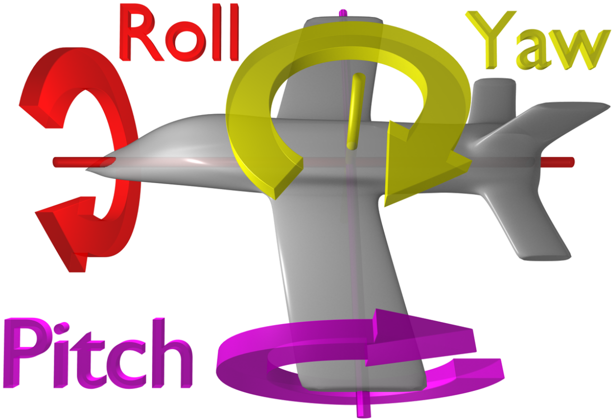
Groups as sets of *actions*

Warning: The following contains extremely mild spoilers for *Legend of Zelda: Tears of the Kingdom*.

Groups as sets of *actions*

In LoZ:TotK, players can translate and rotate objects.

Groups as sets of *actions*



(Ignore the direction of the arrows.)

Groups as sets of *actions*

Rotation by e = identity:

$$\overset{e}{!}$$

Groups as sets of *actions*

Rotation by p (for *pitch*):

$\xrightarrow{p}$

What's the action of p^{-1}?

$\xrightarrow{p^{-1}}$

Groups as sets of *actions*

Rotation by y (for *yaw*):

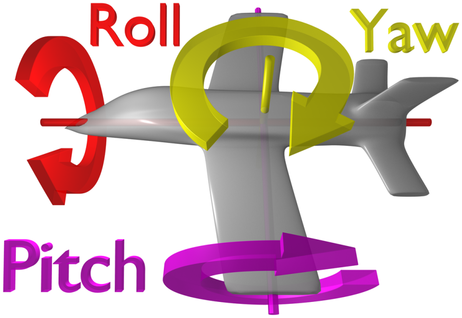
y !

What's the action of $y \ ^{1}$?

$y \ ^{1}$!

Groups as sets of *actions*

The controls only allow you to act by $p; p\ ^{1}; y; y\ ^{1}$.
But what about *roll*?



(Again, ignore the direction of the arrows.)

Groups as sets of *actions*

Question (write answer on participation slip)

Let r denote (counterclockwise) rotation by $\frac{}{4} = 45$ in the *roll* direction, as shown below. How do you perform r, as a sequence of the available actions?

$\overset{r}{!}$

(p) (y)

Matrix groups

A *matrix group* is a group whose elements are matrices.

- The group operation is matrix multiplication.
- Matrix groups provide a wide range of examples, and showcase many different properties that groups can have.
- We often use matrix groups as avatars for more general groups; this is the foundational idea behind *representation theory*.

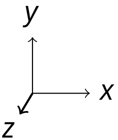
cf. HW00

$M_{2\times 2}(\mathbb{R}) = \{2 \times 2$ matrices with entries in $\mathbb{R}\}$ is a *noncommutative* or *non-abelian* matrix group,
i.e. given $A, B \in M_{2\times 2}$, AB is not always BA.

Linear algebra will feature heavily in this course.

Matrix groups

Let's write down $p, y,$ and r as matrices acting on $\mathbb{R}^3$,
i.e. linear transformations $p, y, r : \mathbb{R}^3 \to \mathbb{R}^3$.



(out of the board)

Matrix groups

$$\xrightarrow{r}$$

To write down how r acts on $\mathbb{R}^3$, we need to compute what it does to the *standard basis vectors*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrix groups

$$\xrightarrow{r}$$

$$r\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad r\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$r\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \text{therefore} \qquad r = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$r = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{r}{=} \begin{pmatrix} \cos\frac{}{4} & \sin\frac{}{4} & 0 \\ \sin\frac{}{4} & \cos\frac{}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark

In general, rotation by along the uv-plane is given by matrix that is $\begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix}$ in the $(u; v)$ coordinates, and the identity matrix everywhere else.

Matrix groups

p!

p acts by rotation by $\ = \frac{}{4}$ in the *zy*-plane (view from the left):

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & \sin & \cos \end{pmatrix}$$

Matrix groups

$$\overset{y}{\to}$$

y acts by rotation by $= \frac{}{4}$ in the xz-plane (view from below):

$$y = \begin{pmatrix} \cos & 0 & \sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{pmatrix}$$

Matrix groups

Let $\quad = \frac{}{4}$.

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & \sin & \cos \end{pmatrix} \qquad y = \begin{pmatrix} \cos & 0 & \sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{pmatrix}$$

$$r = \begin{pmatrix} \cos & \sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise (More details in HW01)

Find matrices representing $p^{\;\;1}$ and $y^{\;\;1}$. Then verify your equation relating r to $p; p^{\;\;1}; y; y^{\;\;1}$.