Lecture 07

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MAT 150A

Participation Slip

1. Take a slip from the front of the room.
2. Write your full name on the top left corner.
3. You will write down your answer to some clearly marked "Participation Slip" questions during lecture.
4. Hand in your slip at the end of class.

Reminder

- Participation slips won't be graded until Lecture 9.
- From Lecture 9 and onward, your participation slip will be graded for completion.
- A score of 15 (out of 20 lecture days) will receive full credit.

Group Homomorphisms

A **homomorphism** $\varphi : G \to G'$ is a map from G to G' such that for all $a, b \in G$,

$$\varphi(ab) = \varphi(a)\varphi(b).$$

The **kernel** of φ is

$$\ker \varphi := \{a \in G \mid \varphi(a) = 1\}.$$

The **image** of φ is

$$\operatorname{im} \varphi := \{x \in G' \mid x = \varphi(a) \text{ for some } a \in G\} =: \varphi(G).$$

Cosets

Let $H \leq G$, and let $a \in G$.

- If written in multiplicative notation, the **left coset** of H containing a is

$$aH = \{g = ah \mid h \in H\}$$

- If G is abelian and written in additive notation, the coset of H containing a is

$$a + H = \{g = a + h \mid h \in H\}.$$

Proposition

Let $\varphi : G \to G'$ be a homomorphism, and let $a, b \in G$. Let $K = \ker \varphi$. The following conditions are equivalent (TFAE):

- $\varphi(a) = \varphi(b)$
- $a^{-1}b \in K$
- $b \in aK$
- $bK = aK$.

Kernels and Cosets

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Corollary

A homomorphism $\varphi : G \to G'$ is *injective* (as a set map) if and only if its kernel K is the trivial subgroup $\{1\}$ in G.

Equivalence Relations and Partitions

To better understand cosets, we need to recall the notations of **equivalence relations** and **partitions**.

$3\mathbb{Z}$	0	3	6	9	12	15	$\cdots$
$1+3\mathbb{Z}$	1	4	7	10	13	16	$\cdots$
$2+3\mathbb{Z}$	2	5	8	11	14	17	$\cdots$

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A **partition** P of a set S is a **collection** $P = \{P_\alpha\}_{\alpha \in I}$ of subsets of S such that

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In other words, $S = \coprod_{\alpha \in I} P_\alpha$, the **disjoint union** of the sets P_α.

Equivalence Relations

Recall that a **relation** R on a set S is a subset of $S \times S$.

- Relations are more general than functions.
- We usually write $a \sim b$. A priori, this is different from saying $b \sim a$ (since in general, $(a, b) \neq (b, a)$ in $S \times S$).

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An **equivalence relation** on a set S is a relation $\sim$ on elements of S that is

1. **reflexive**: For all a, $a \sim a$.
2. **symmetric**: If $a \sim b$, then $b \sim a$.
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Participation Slip

Let a, b be elements of a group G.
We say a is **conjugate to** b if there exists $g \in G$ such that

$$b = gag^{-1}.$$

Prove that **conjugacy** is an equivalence relation.

Equivalence Relations and Partitions

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Definition

For every equivalence relation on a set S, there is a surjective map

$$\pi : S \to \bar{S} \qquad a \mapsto \bar{a}$$

that maps each element $a \in S$ to its equivalence class $\bar{a}$. Here $\bar{S}$ denotes the set of equivalence classes.

Example: $\mathbb{Z}, 3\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z}$

Define an equivalence relation on $\mathbb{Z}$ as follows:

$$m \sim n \qquad \text{iff} \qquad m \equiv n \mod 3.$$

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There is a surjective map (currently only a set map, but actually a homomorphism)

$$\pi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$$
$$n \mapsto \bar{n}.$$

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- (Symmetry) If $a \sim b$, then $b = ah$ for some h, so $a = bh^{-1}$; since H is a subgroup, $h^{-1} \in H$ as well, so $b \sim a$.

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- (Reflexivity) $a = a1$ and $1 \in H$ so $a \sim a$.
- (Symmetry) If $a \sim b$, then $b = ah$ for some h, so $a = bh^{-1}$; since H is a subgroup, $h^{-1} \in H$ as well, so $b \sim a$.
- (Transitivity) If $a \sim b$ and $b \sim c$, then there exist h, h' such that $b = ah$, $c = bh'$. Then $c = bh' = (ah)h' = a(hh')$ where $hh' \in H$ (again because H is a subgroup), so $a \sim c$.

□

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Proof.

The obvious map $(g\cdot) : H \to gH$ defines a bijection, because it has an inverse $(g^{-1}\cdot) : gH \to H$. □