Lecture 14

Melissa Zhang

MAT 150A

Participation Slip

1. Take a slip from the front of the room.
2. Write your full name on the top left corner.
3. You will write down your answer to some clearly marked "Participation Slip" questions during lecture.
4. Hand in your slip at the end of class, in the pile according to the first letter of your surname.

Today: The Correspondence Theorem (§2.10 in the text)

We will build up concepts, notation, terminology, and propositions to prove the following theorem:

Theorem 2.10.5: Correspondence Theorem

Let $\varphi : G \to \mathcal{G}$ be a surjective group homomorphism with kernel K. Then there is a bijective correspondence

$$\{\text{subgroups of } G \text{ that contain } K\} \leftrightarrow \{\text{subgroups of } \mathcal{G}\}.$$

Restriction Maps

Let $\varphi : G \to \mathcal{G}$ be a group homomorphism, and let $H \leq G$.
We may **restrict** φ to a homomorphism

$$\begin{aligned} \varphi|_H : H &\to \mathcal{G} \\ h &\mapsto \varphi(h) \end{aligned}$$

- $\ker(\varphi|_H) = (\ker \varphi) \cap H$
- $\operatorname{im}(\varphi|_H) = \varphi(H)$

Restriction Maps

Observation Since $\varphi|_H$ is a homomorphism, the order of the image $\varphi(H)$ divides both $|H|$ and $|\mathcal{G}|$. If $|H|$ and $|\mathcal{G}|$ have no common factors, then $H \leq \ker \varphi$.

Example

Recall A_n is the kernel of the sign homomorphism $\sigma : S_n \to \pm 1$. Let q be a permutation with odd order, and let $H = \langle q \rangle$. Then $H \leq A_n$.

Proposition 2.10.4

Let $\varphi : G \to \mathcal{G}$ be a homomorphism with kernel K. Let $\mathcal{H} \leq \mathcal{G}$, and let $H = \varphi^{-1}(\mathcal{H})$.

1. Then $K \leq H \leq G$. (A chain of subgroups.)
2. If $\mathcal{H} \trianglelefteq \mathcal{G}$, then $H \trianglelefteq G$.
3. If φ is surjective and $H \trianglelefteq G$, then $\mathcal{H} \trianglelefteq \mathcal{G}$.

Example

Consider $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$. Since $\mathbb{R}^\times$ is abelian, $\mathbb{R}^\times_{>0} \trianglelefteq \mathbb{R}^\times$. The preimage under det of the positive reals is the set of invertible matrices with positive determinant, and is therefore a normal subgroup of $GL_n(\mathbb{R})$.

Proposition 2.10.4 (summarize on side board)

Let $\varphi : G \to \mathcal{G}$ be a homomorphism with kernel K. Let $\mathcal{H} \le \mathcal{G}$, and let $H = \varphi^{-1}(\mathcal{H})$.

1. Then $K \le H \le G$. (A chain of subgroups.)
2. If $\mathcal{H} \trianglelefteq \mathcal{G}$, then $H \trianglelefteq G$.
3. If φ is surjective and $H \trianglelefteq G$, then $\mathcal{H} \trianglelefteq \mathcal{G}$.

Proof.

1. Check carefully; note that φ^{-1} means preimage.
2. Suppose $\mathcal{H} \trianglelefteq \mathcal{G}$. Let $x \in H, g \in G$. Then $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} \in \mathcal{H}$ because $\mathcal{H} \trianglelefteq \mathcal{G}$.
3. Suppose φ is surjective and $H \trianglelefteq G$. Let $a \in \mathcal{H}, b \in \mathcal{G}$. Since φ is surjective, there exist elements $x \in H, y \in G$ such that $\varphi(x) = a$, $\varphi(y) = b$. Since H is normal, $yxy^{-1} \in H$, so $\varphi(yxy^{-1}) = bab^{-1} \in \mathcal{H}$.

The Correspondence Theorem

Theorem 2.10.5: Correspondence Theorem

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$$\{\text{subgroups of } G \text{ that contain } K\} \leftrightarrow \{\text{subgroups of } \mathcal{G}\}.$$

The correspondence is given by

$$\mathcal{H} \rightsquigarrow \varphi^{-1}(\mathcal{H}).$$

Suppose H and $\mathcal{H}$ are corresponding subgroups. Then:

- $H \trianglelefteq G$ if and only if $\mathcal{H} \trianglelefteq \mathcal{G}$.
- $|H| = |\mathcal{H}||K|$.

Can you see why this correspondence would be very useful?

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Participation Slip

There are about five statements here that we have to check (depending on how you chunk them). What are they?

This is in the book; try not to look at the book when attempting this exercise.

Things to check

We use the notation already introduced in this section.

1. $\varphi(H)$ is a subgroup of $\mathcal{G}$
2. $\varphi^{-1}(\mathcal{H})$ is a subgroup of G, and it contains K
3. $\mathcal{H} \trianglelefteq \mathcal{G}$ if and only if $\varphi^{-1}(\mathcal{H}) \trianglelefteq G$
4. Bijectivity of the correspondence: $\varphi(\varphi^{-1}(\mathcal{H})) = \mathcal{H}$ and $\varphi^{-1}\varphi(H) = H$.
5. $|\varphi^{-1}(\mathcal{H})| = |\mathcal{H}||K|$.