Lecture 15

Melissa Zhang

MAT 150A

Don't forget to pick up a participation slip!

Quotient Groups: Main Idea

The cosets of a normal subgroup form a group.

- The set of cosets of $N \trianglelefteq G$ is denoted $\bar{G} = G/N$.
  - Why am I not writing "left cosets" or "right cosets" here?
- The coset containing $a \in G$, may be written as

$$aN,\ \bar{a},\ \text{or}\ [a].$$

  - Recall that the latter two notations are fairly standard for the notion of equivalence class.
- There is a **canonical** (i.e. obvious) map (of sets, currently)

$$\pi : G \to \bar{G}, \qquad g \mapsto \bar{g}.$$

  - Clearly, π is a surjective map, and $\pi^{-1}(\bar{1}) = N$.

Additive example: $(\mathbb{Z}/5\mathbb{Z}, +)$

$G = \mathbb{Z}$ is an abelian group, so $N = 5\mathbb{Z} \trianglelefteq \mathbb{Z}$.
The set of cosets of N are $\bar{G} = G/N = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\} = \mathbb{Z}/5\mathbb{Z}$.

- $\bar{3} = 3 + 5\mathbb{Z} = [3] \in \bar{G}$

Multiplicative example: Product groups

Let $G \cong H \times K$, where $H, K \leq G$. Recall that this means $H \times K \to HK = G$ is an isomorphism.

Then $K \trianglelefteq G$, and the set of cosets of K are

$$\{hK \mid h \in H\}$$

so the quotient group is a copy of H.

Under the isomorphism $G \to H \times K$, the coset hK (for a fixed h) corresponds to the set of pairs $\{(h, k) \mid k \in K\}$.

Quotient Group Structure

Participation Slip

1. What would be the most obvious multiplication operation for $\bar{G}$?
2. Why is it therefore necessary for N to be normal in G for $\bar{G}$ to form a group?

1. $[a][b] = [ab]$, i.e. $(aN)(bN) = (ab)N$, i.e. $\bar{a}\bar{b} = \overline{ab}$
2. $aNbN = a(bN)N = abN$ This is Lemma 2.12.5 in the text.

Aside

Do you understand the meaning of the words **necessary** and **sufficient** in mathematics?

Quotient Group Structure

Theorem 2.12.2

Let $N \trianglelefteq G$, and let $\bar{G}$ denote the set of cosets of N in G.

- There is a law of composition on $\bar{G}$ that makes it into a group.
- Using that law of composition, the canonical map $\pi : G \to \bar{G}$ defined by $g \mapsto \bar{g}$ is a surjective homomorphism whose kernel is N.

To prove:

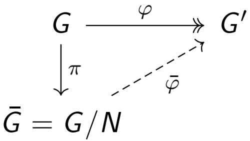
1. $[a][b] = [ab]$ makes $\bar{G}$ into a group
2. π is a surjective homomorphism
3. $\ker \pi = N$

Quotient Group Structure

1. $[a][b] = [ab]$ makes $\bar{G}$ into a group
   - Associativity is easily verified:
     $(\bar{a}\bar{b})\bar{c} = \overline{ab}\bar{c} = \overline{abc} = \bar{a}\overline{bc} = \bar{a}(\bar{b}\bar{c})$
   - Also easily verified: $[1]$ is the identity, and $[a]^{-1} = [a^{-1}]$
2. π is a surjective homomorphism
   - π is surjective as a set map; it remains to check $\pi(a)\pi(b) = \pi(ab)$; but this is precisely the definition of multiplication in $\bar{G}$
3. $\ker \pi = N$
   - $\pi^{-1}([1]) = N$ indeed

$\bar{G}$ was defined so that $\pi : G \to \bar{G}$ would be a surjective homomorphism.

First Isomorphism Theorem

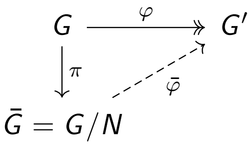


First Isomorphism Theorem

Let $\varphi : G \to G'$ be a surjective homomorphism with kernel $N \trianglelefteq G$. Then φ **factors through** G/N, and there is a unique isomorphism $\bar{\varphi} : G/N \to G'$ such that $\varphi = \bar{\varphi} \circ \pi$.

Exercise: Prove that $\bar{\varphi}$ is indeed determined by π and φ, and therefore must be unique.

First Isomorphism Theorem



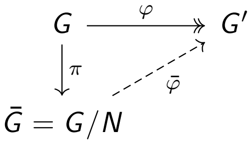
First Isomorphism Theorem

Let $\varphi : G \to G'$ be a surjective homomorphism with kernel $N \trianglelefteq G$. Then φ **factors through** G/N, and there is a unique isomorphism $\bar{\varphi} : G/N \to G'$ such that $\varphi = \bar{\varphi} \circ \pi$.

Proof. Let $\bar{\varphi}(\bar{x}) = \varphi(x)$.

- Check that $\bar{\varphi}$ is **well-defined**.
- Check that $\bar{\varphi}$ is a homomorphism.

Then $\bar{\varphi} \circ \pi(x) = \varphi(x)$ by construction.



First Isomorphism Theorem

Let $\varphi : G \to G'$ be a surjective homomorphism with kernel $N \trianglelefteq G$. Then φ **factors through** G/N, and there is a unique isomorphism $\bar{\varphi} : G/N \to G'$ such that $\varphi = \bar{\varphi} \circ \pi$.

Corollary

Let $\varphi : G \to G'$ be any (not necessarily surjective) group homomorphism. Then

$$G/\ker\varphi \cong \operatorname{im}\varphi.$$

Examples

If $\varphi : G \to G'$ is surjective, then $G' \cong G/\ker\varphi$.

- What is $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ isomorphic to?
  - A: Consider $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$. It's surjective. Since $\ker(\det) = \mathrm{SL}_n(\mathbb{R})$, the quotient group is isomorphic to $\mathbb{R}^\times$.
- What is $\mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C})$ isomorphic to?
  - A: The homomorphism $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times$ is surjective, with kernel $\mathrm{SL}_n(\mathbb{C})$. Therefore $\mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C}) \cong \mathbb{C}^\times$.
- Identify the quotient group $\mathbb{C}^\times/S^1$.
  - Recall that the multiplicative group of complex numbers $\mathbb{C}^\times$ contains the circle group S^1 consisting of all the complex numbers of modulus 1.
  - A: Consider the absolute value map $|\cdot| : \mathbb{C}^\times \to \mathbb{R}^\times_{>0}$. It's surjective. Then $S^1 = \ker(|\cdot|)$. So $\mathbb{C}^\times/S^1 \cong \mathbb{R}^\times_{>0}$.