Lecture 17

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MAT 150A

Don't forget to pick up a participation slip!

Today

- Cycle types and conjugacy classes in S_n (10am class)
- Symmetries and group actions
  - intuition from §6.1, 6.2
  - back to groups with §6.4
- 10 minutes before end of class: return exams. (You are encouraged to remind me to stop talking when it's 10 minutes before the end of class.)

Intuitive Notion of “Symmetry”

Groups were invented to describe symmetries of all sorts of mathematical objects, not just things we can visualize.

To get a better sense of **group actions**, we’ll first use our geometric intuition to look at some symmetries of figures in the plane.

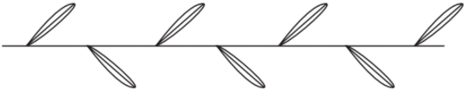
Symmetry of Plane Figures

Bilateral Symmetry.

Rotational Symmetry.

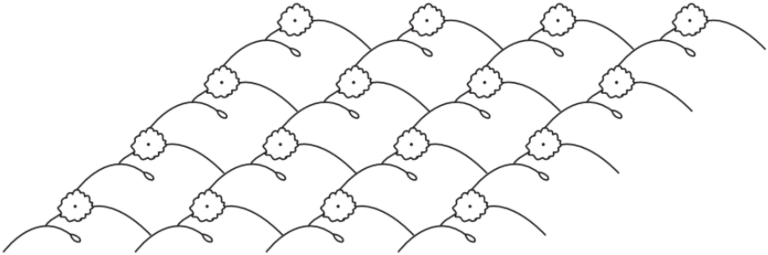
Symmetry of Plane Figures

Translational Symmetry.



Glide Symmetry.

Symmetry of Plane Figures



Isometries of the Plane

- A *rigid motion* of the plane is called an **isometry** of the plane.
- Isometries preserve **distance**.
  - Recall that *metric* = *notion of distance*.
  - Given any metric space X, the group of isometries of X is the set of all maps $\varphi : X \to X$ such that the metric is preserved.
- Isometries of the plane $\mathbb{R}^2$:
  - Orthogonal linear transformations: $\varphi(x) = Px$ where $P \in O(2)$.
  - Translations: $\varphi(x) = x + v$.

Symmetries of Subsets of the Plane

In this class, "plane figure" means "subset of the plane".

Let $F \subset \mathbb{R}^2$ be a subset of the plane. If an isometry of the plane takes F to itself, **set-wise**, then it is called a **symmetry of F.**

- *Set-wise* means that for all $p \in F$, $\varphi(p) \in F$.
- This is as opposed to a function φ that fixes points in a set *point-wise*, i.e. for all $p \in F$, $\varphi(p) = p$.

Orientations

An isometry of the plane $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is **orientation-preserving** if it takes the "front" to the "front". Otherwise, it is **orientation-reversing**.

- An **orientation** on $\mathbb{R}^2$ is a choice of an ordered pair of orthogonal vectors, such as (e_1, e_2), *modulo* or *up to* rotation.
- To determine whether φ orientation-preserving or orientation-reversing, check whether $\varphi(e_2)$ is 90° counterclockwise from $\varphi(e_1)$.
- This is the same as checking the "right-hand rule" from physics, or checking the sign of the cross product, or checking the determinant of a related matrix.

The Takeaway from §6.1, 6.2

The group of isometries of the plane $\mathbb{R}^2$ is *generated* by

1. **translation** t_a by a vector a:

$$t_a(x) = x + a = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

2. **rotation** ρ_θ by an angle θ about the origin:

$$\rho_\theta(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. **reflection** r about the e_1-axis:

$$r(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Participation Slip

Let r' denote reflection about the e_2-axis. Find a word in the generators above that is equal to r'.

Back to Groups (§6.4)

The group of isometries of the plane $\mathrm{Isom}(\mathbb{R}^2)$ is generated by

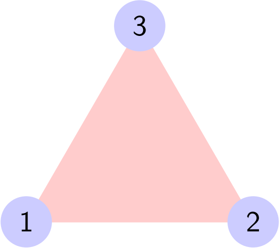
- **rotation and reflection: these generate** $O(2)$
- translation

If F is a plane figure, then its group of symmetries will be a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$. the group of isometries of plane figures will be a subgroup of the above.

Dihedral Groups

The **dihedral group** D_n (also denoted D_{2n} by others) is the group of symmetries of a **regular** n**-gon** centered at the origin.
Hence $D_n \leq O(2)$, and is therefore generated by rotations and reflections.

D_3 (or D_6): Symmetries of the Equilateral Triangle

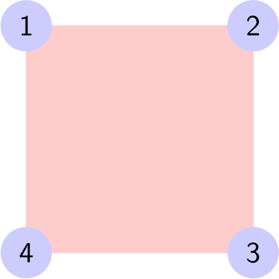


- Any symmetry of the triangle will permute the vertices, and is determined by how the vertices are determined. Do you agree with this geometric claim?
- Therefore $D_3 \leq S_3$.

Observe that any permutation of the vertices is a valid symmetry of the triangle. Therefore $D_3 = S_3$.

To worried philosophers out there: Here we define S_3 here not as the Platonic ideal of S_3, but as the group that acts on the vertices. I would normally write $D_3 \cong S_3$.

D_4 (or D_8): Symmetries of the Square



- Again, symmetries are permutations of vertices, so $D_3 \leq S_4$.
- But not all permutations in S_4 give valid symmetries of this rigid square. Can you think of a permutation that isn't a symmetry?

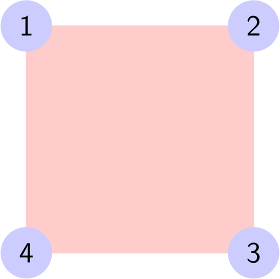
Geometric Reasoning: $|D_4| = 8$

All the symmetries of the square are of the form

- rotate by a multiple of $\frac{\pi}{2}$ or
- flip to the back side and then rotate by a multiple of $\frac{\pi}{2}$.

These are just the cosets of the normal subgroup of rotations!

D_4 (or D_8): Symmetries of the Square

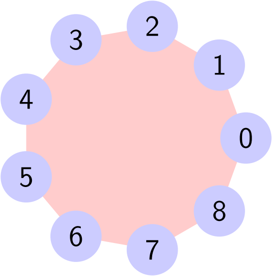


- Let $\rho =$ rotation by $\frac{\pi}{2}$ CCW, and $\tau =$ reflection about the e_1-axis.
- Clearly $\rho^4 = \tau^2 = 1$. There is another relation, which we can already see in $O(2)$: $\tau\rho\tau = \rho^{-1}$.
- $D_4 = \langle \rho, \tau \mid \rho^4 = \tau^2 = \tau\rho\tau\rho = 1 \rangle$

The 8 elements of D_4 are

$$\{1, \rho, \rho^2, \rho^3\} \sqcup \{\tau, \tau\rho, \tau\rho^2, \tau\rho^3\}.$$

D_n (or D_{2n}): Symmetries of the Regular n-gon



- Let $\rho =$ CCW rotation by $\frac{2\pi}{n}$ CCW, and $\tau =$ reflection about the e_1-axis.
- Clearly $\rho^n = \tau^2 = 1$. We still have the relation $\tau\rho\tau = \rho^{-1}$.
- $D_n = \langle \rho, \tau \mid \rho^n = \tau^2 = \tau\rho\tau\rho = 1 \rangle$

The $2n$ elements of D_n are

$$\{1, \rho, \ldots, \rho^{n-1}\} \sqcup \{\tau, \tau\rho, \ldots, \tau\rho^{n-1}\}.$$

Exam 1 Return

- I will call out your name; please raise your hand and correct my pronunciation if you'd like to.
- Check solutions posted on the class website.
- If you have questions about your exam, go to TA office hours.