Lecture 19

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MAT 150A

Don't forget to pick up a participation slip!

Reminders

- No class this Friday. Veteran's Day.
- Today: §6.5, discrete groups of isometries
- Exam 2 is next Wednesday, in class. A study guide will be posted on the class website later this week.

Recall: Isometries of $\mathbb{R}^2$

The group of isometries of the plane $\mathbb{R}^2$, denoted $\mathrm{Isom}(\mathbb{R}^2)$, is *generated* by

1. *translation* t_a by a vector a
2. *rotation* ρ_θ by an angle θ about the origin
3. *reflection* r about the e_1-axis

All isometries of the plane are of the following three types:

1. **translation**
2. operation by an element of $O(2)$ (**rotation** or **reflection**)
3. a **glide reflection**: reflection about a line ℓ, followed by translation by a nonzero vector parallel to ℓ

Keep these three in mind.

Isometries of Real Vector Spaces

Let $\mathbb{V}$ be a real vector space and let $\langle \cdot, \cdot \rangle$ be a **metric** on $\mathbb{V}$:

- This is a generalization of the dot product on our classical Euclidean $\mathbb{R}^2$.

Recall that $\mathbb{V}$ has a group structure, under vector addition.

Notation/Definition

The group of isometries of $(\mathbb{V}, \langle \cdot, \cdot \rangle)$, denoted $\mathrm{Isom}(\mathbb{V})$, is the set of isomorphisms $g : \mathbb{V} \to \mathbb{V}$ that preserve the metric:

$$\langle g(v), g(w) \rangle = \langle v, w \rangle.$$

Participation Slip

Prove that this implies distance is preserved.

Isometries of Real Vector Spaces

Warning

There are two types of groups we are talking about today:

- The vector space $\mathbb{V}$, with vector addition.
- The group of isometries $\mathrm{Isom}(\mathbb{V})$ acting on $\mathbb{V}$.

We will talk about subgroups of each of these.

- We will use capital Greek letters, such as Λ, for subgroups of $\mathbb{V}$.
- We will use capital Roman letters, such as T, for subgroups of $\mathrm{Isom}(\mathbb{V})$.

Make sure you are clear on the difference between these two types of subgroups.

Fact

The n-dimensional Euclidean space $\mathrm{Isom}(\mathbb{R}^n)$ is also comprised of

- translations $T = \{t_v \mid v \in \mathbb{V}\}$
- actions by elements of the orthogonal group $O(\mathbb{V})$ ($\cong O(n)$)
- operations of $O(\mathbb{V})$ on translations

The Homomorphism $\mathrm{Isom}(\mathbb{V}) \to O(\mathbb{V})$

There is a homomorphism $\varphi : \mathrm{Isom}(\mathbb{V}) \to O(\mathbb{V})$ with $\ker \varphi = T$, the subgroup of translations.

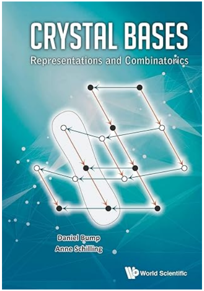
$L \trianglelefteq \mathrm{Isom}(\mathbb{R}^n)$, $\mathrm{Isom}(\mathbb{R}^n) = L \rtimes O(\mathbb{R}^n)$

Our Goal

If we wish to tile a wall, what types of patterns are possible?

Why is this important?

- pretty patterns, quirky bathroom floors (see Prof. Kuperberg's website)
- Crystallographic groups show up in algebraic combinatorics, and are related to the representation theory of Lie groups
- **Lattices** and **point groups** are independently interesting.

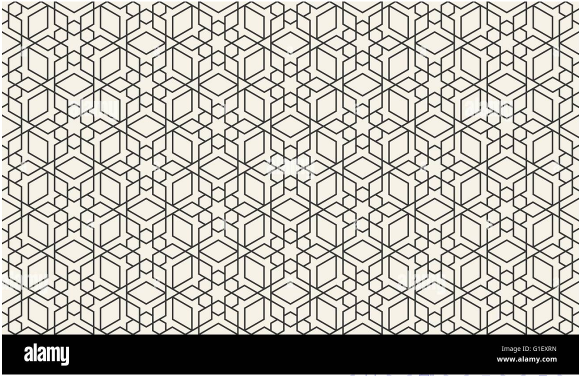


Book by Bump and Professor Anne Schilling

Our Goal

If we wish to tile a wall, what types of patterns are possible?

- the pattern should be periodic, i.e. exhibit symmetry
- the tiles must be a reasonable size (i.e. not infinitesimally small)



The Crystallographic Restriction

For the rest of this lecture and the next, you may assume that $\mathbb{V} \cong \mathbb{R}^2$, with a metric $\langle \cdot, \cdot \rangle$, which we may also write as a dot product $\cdot$.

Theorem 6.5.12: Crystallographic Restriction

Let Λ be a discrete subgroup of $\mathbb{V}$, and let $\mathrm{Sym}(\Lambda)$ denote the group of symmetries of Λ.
Let $H \subset O(2)$ a subgroup of $\mathrm{Sym}(\Lambda)$. E.g. $H = O(2) \cap \mathrm{Sym}(\Lambda)$
Suppose that $\Lambda \neq \{\mathbf{0}\}$, the trivial subgroup. Then:

1. every rotation in H has order 1, 2, 3, 4, or 6, and
2. H is one of the groups C_n or D_n, where $n \in \{1, 2, 3, 4, 6\}$.

Discrete Groups of Isometries

Definition

A group G of isometries of the plane $\mathbb{R}^2$ is discrete if it does not contain arbitrarily small translations or rotations.

- For reference: A group H *does contain* an arbitrarily small translation if
  for any $\varepsilon > 0$, there is a translation $t_v \in H$ such that $0 < |v| < \varepsilon$.

So, we can reformulate the definition as follows:

G is discrete if there exists a real number ε so that

- if $t_v \in G$ and $v \neq 0$ (i.e. $t_v \neq \text{id}$), then $|v| \geq \varepsilon$, and
- if $\rho\theta \in G$, where $\theta \in [-\pi, \pi)$, then $|\theta| \geq \varepsilon$.

There are three main tools for analyzing a discrete group G:

1. the **translation (sub)group** $L \leq G$, a subgroup of the group T of translation vectors in $\mathrm{Isom}(\mathbb{V})$
2. the **point group** $\overline{G}$, a subgroup of the orthogonal group $O(2)$
3. an operation of $\overline{G}$ on L (glide reflection or glide)

The Translation Group

Note that every discrete subgroup L of T is also a discrete subgroup Λ of $\mathbb{V}$ itself. Why is this?

Theorem

Every discrete subgroup Λ of $\mathbb{V}$ is one of the following:

a. *the zero group: $L = \{\mathbf{0}\}$*
b. *the set of integer multiples of a nonzero vector a:*

$$L = \mathbb{Z}a = \{ma \mid m \in \mathbb{Z}\}$$

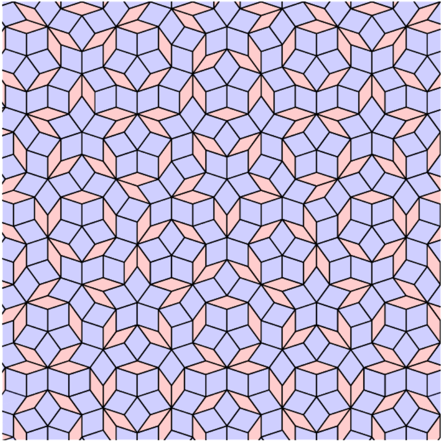
c. *the set of integer combinations of two linearly independent vectors a and b:*

$$L = \mathbb{Z}a + \mathbb{Z}b = \{ma + nb \mid m, n \in \mathbb{Z}\}$$

Groups of this type are called **lattices.**

(See sketch of proof on board, or full proof in the textbook.)

...The Penrose Tiling?



Source: Felix Flicker, Steven Simon, and S. A. Parameswaran, (2020). Classical Dimers on Penrose Tilings. Physical Review X. 10. 10.1103/PhysRevX.10.011005.