LECTURE 14

No participation slips until further notice. (We're not an evening class, but some students commute and it makes no sense to come for fewer classes.) Stay safe!

<u>Recall</u> Two types of elem. orthogonal matrices (rigid transformations)

- Givens rotations: (rotation matrices)

$$G = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & c & & s & \\ & & & & 1 & & \\ & & & -s & & c & \\ & & & & & & 1 \end{bmatrix}$$

i.e. under a permutation of bases, this is just id on all components except the two spanning a particular plane; then do plane rotation there.

- Householder trans (reflection matrices)

$$H = I - 2\frac{vv^T}{v^Tv} = I - 2 \cdot \frac{v}{\|v\|} \cdot \frac{v^T}{\|v\|} = I - 2\underbrace{uu^T}$$

orthogonal projector P

$\Rightarrow$ $I - 2P$ is an orthogonal matrix (HW04).

Projection with orthonormal basis

Suppose you have a subspace given as the span of an orthonormal basis: $\quad \text{span}\{q_1, \dots, q_n\} \subset \mathbb{R}^m$.

Let $Q = [q_1 \; \dots \; q_n]$

$$m\left\{\underbrace{\begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix}}_{n}\right.$$

Then QQ^T is an orthogonal projector too!

$$\begin{bmatrix} q_1 \; Q \; q_n \end{bmatrix}\begin{bmatrix} \leftarrow q_1 \rightarrow \\ Q^T \\ \leftarrow q_n \rightarrow \end{bmatrix}\overset{v}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = \underbrace{[q_1 \cdots q_n]\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}}\, v$$

$$= \underbrace{\left(\sum_{i=1}^{n} q_i q_i^T\right)}_{\text{sum of rank 1 } m\times m \text{ matrices!}} v$$

The complementary projector is also orthogonal (clear from defn!)

Projection with arbitrary basis (related to something familiar...)

Now let $\{a_1, \dots, a_n\}$ be an arbitrary basis for our subspace we want to project onto.

Organize as $A = [a_1 \dots a_n] \in \mathbb{R}^{m \times n}$ (recall $n \leq m$)

* Note that A is full (column) rank.

Let $v \in \mathbb{R}^m$. Then $v = v_1 + v_2$ where $v_1 \in \text{range}(A)$, $v_2 \in \text{null}(A^T)$

$\Rightarrow v_1 = Ax$ for some $x \in \mathbb{R}^n$. (domain of A is $\mathbb{R}^n$)

"$\forall i,\ a_i \perp v_2 = v - v_1 = v - Ax$" (want orthogonal projection)

iff $a_i^T(v - Ax)$ for all i

iff $A^T(v - Ax) = 0.$ ⇝ ?

iff $A^TAx = A^Tv$ the normal eqns!

recall we had wanted to project onto the range of a particular matrix. This is exactly what we were doing.

Q. What is the actual projector in the normal eqns? (Where are we projecting?
A: onto range(A))

A. We wanted to project away the residual.

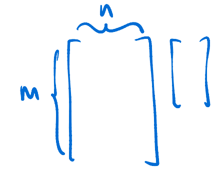
$A^TAx = A^Tb$ Wanted to solve for x (the parameters of our model) and I claimed it was the image of b (v here) under projection.

So solve: (recall A full rank $\Rightarrow A^TA$ invertible)

$x = (A^TA)^{-1}A^Tb$ We want to project b onto range A.

$$\Rightarrow P_A(b) = Ax = A(A^TA)^{-1}A^Tb \Rightarrow \boxed{P_A = A(A^TA)^{-1}A^T}$$

<u>Aside</u> Pseudo inverses. Let $A \in \mathbb{R}^{m\times n}$. $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$



If A (as above) isn't square, then there is no "A^{-1}".

However, if we ask for a matrix B such that

$B\cdot A = I$ and $A \cdot B = I$ ...

- $BA = I_n$ since domain$(A) = \mathbb{R}^n$
- $AB = I_m$ since codomain$(A) = \mathbb{R}^m$

} the dream.

* also clear here that A must be full column rank! Otherwise not an injective function $\Rightarrow$ can't possibly compose with B to be injective.

So assume A is full column rank. (as in normal equations)

$\Rightarrow$ A^TA <u>is</u> invertible. Let $B = (A^TA)^{-1}A^T$

Then: $\cdot$ $BA = (A^TA)^{-1}A^T\cdot A = I = I_n$ indeed!

$\cdot$ $AB = A(A^TA)^{-1}A^T$ ... well, this is the best we got.

if A were square, and thus invertible, we would have access to "A^{-1}" and this would really be I_m.

<u>defn.</u> The pseudo inverse A^+ of a full rank matrix $A \in \mathbb{R}^{m\times n}$ (ie. $n \le m$) is $A^+ = (A^TA)^{-1}A \in \mathbb{R}^{n\times m}$

Compare with

" $\Rightarrow P_A(b) = Ax = A(A^TA)^{-1}A^Tb \Rightarrow \boxed{P_A = A(A^TA)^{-1}A^T}$ "

ie $P_A = A^+A$.

<u>Rmk</u> On the subspace range$(A) \subset \mathbb{R}^m$, P_A really is the identity:

If $v = Ax$, then $P_A v = v$ (recall: $P_A v - v = P_A x - P_A x = 0$)

And so $P_A|_{\text{range}(A)} = \mathrm{Id}_{\text{range}(A)}$ dim range$A = n$; $P_A|_{\text{range}A} \sim I_n$.