MAT167 HW06

Due 5/18/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Covered material This homework covers algorithms for QR decomposition: Classical Gram-Schmidt (CGS) with partial pivoting, Modified Gram-Schmidt (MGS), and Householder triangularization (Chapter 5 in the book).

Exercise 1

The goal of this exercise is to understand why MGS is "triangular orthogonalization" resulting in a thin QR decomposition.

Consider the following 4×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

(a) Run the MGS algorithm on A by hand using the variables we used in class (column vectors are a_j, $v_j^{(k)}$ is the vector $v_j^{(1)} = a_j$ updated $k-1$ times, r_{ij} are the entries of the R matrix, etc.). Report the resulting $\hat{Q}$ matrix in terms of the a_{ij} and r_{ij}.

(b) Referring to your work in part (a), for each of the two steps, explicitly write down the operations you have performed on the matrix A as a matrix-matrix multiplication, i.e. explicitly write out AR_1 and $(AR_1)R_2$, respectively, as products of two matrices, in terms of the a_{ij} and r_{ij} variables. Multiply out your matrices to obtain the explicit matrices $\begin{bmatrix} q_1 & v_2^{(2)} \end{bmatrix}$ and $\begin{bmatrix} q_1 & q_2 \end{bmatrix}$, respectively.

Exercise 2

Let

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$

where ϵ is a small positive number (e.g., 10^{-7}) so that ϵ^2 can be ignored numerically.

(a) Compute the reduced QR factorization $A = \widehat{Q}\widehat{R}$ using the classical Gram-Schmidt algorithm by hand.

(b) Compute the reduced QR factorization $A = \widehat{Q}\widehat{R}$ using the modified Gram-Schmidt algorithm by hand.

(c) Compute the full QR factorization $A = QR$ using the Householder triangularization by hand.

(d) Check the quality of these results by computing the Frobenius norm of $\|\widehat{Q}^\top \widehat{Q} - I\|_F$ for the results obtained by the CGS and MGS algorithms and $\|Q^\top Q - I\|_F$ for the result obtained by the Householder triangularization.

Exercise 3

This exercise will be graded for completion.

The goal of this exercise is to survey the class for interest in various final project topics. Once I choose the most popular topics (∼3 different projects), you will be able to choose which project to complete for your final exam.

Your final project will be a report on an exploration of a particular application of the matrix methods we studied in class. The last decomposition we will study is singular value decomposition. Afterwards, we'll move on to the applications in the textbook.

Your final project will be related to a topic in a different textbook, by Moler: Numerical Computing with MATLAB. You will study a particular application of the matrix methods we learned in class by answering a series of problems similar to some homework problems.

Scroll through the following chapters of Moler's book and pick **two applications** that are of greatest interest to you, and that you would be happy to focus on for your final project. Write down the Chapter number and the page of the PDF you are referring to for each topic. **Submit this as the solution to Exercise 3 on this HW06.**

- Chapter 2: Linear Equations [PDF]
- Chapter 5: Least Squares [PDF]
- Chapter 8: Fourier Analysis [PDF] (We obviously didn't discuss Fourier analysis in this course; choose a topic from this Chapter only if you would be comfortable reading this chapter; e.g., you would need to be at the very least comfortable with complex numbers.)
- Chapter 10: Eigenvalues and Singular Values [PDF]

More information about the final exam:

- All materials for the final project for this course will be due at **11:59 pm on the last day of class, June 8, 2023, on Gradescope**. You cannot use your late tokens on the final project. However, as this assignment is very important, I will still accept final project submissions up until *11:59 pm on June 9, 2023*; this is a *hard deadline*.
- Your finished product must be **typeset**, using whatever typesetting software you like (e.g. LaTeX, MS Word, Google Docs). Your final product must be **submitted as a PDF to Gradescope**.
- You will be graded on mathematical accuracy, content, clarity, presentation, among other aspects. A rubric will be provided to you with more details.