MAT167 HW07

Due 5/25/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Covered material This homework covers SVD, Chapter 6.

Exercise 1

Let $A \in \mathbb{R}^{m\times n}$, and suppose $B \in \mathbb{R}^{n\times m}$ is obtained by rotating A 90 degree clockwise on paper. More precisely,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} a_{m1} & \cdots & a_{21} & a_{11} \\ \vdots & \vdots & \vdots & \vdots \\ a_{mn} & \cdots & a_{2n} & a_{1n} \end{bmatrix}.$$

Do A and B have the same singular values? Prove that the answer is yes or give a counterexample. [Hint: Express B as a product of $A^\top$ and a matrix P that permutes the column vectors of $A^\top$.]

Exercise 2

Two matrices $A, B \in \mathbb{R}^{m\times m}$ are *orthogonally equivalent* if $A = QBQ^\top$ for some orthogonal matrix Q. Is it true or false that A and B are orthogonally equivalent if and only if they have the same singular values?

Exercise 3

Using MATLAB, do the following:

(a) Load the image called `mandrill.mat`, via:

```
>> load mandrill;
```

This loads a matrix X containing the face of a mandrill, and a map containing the colormap of that image. If you cannot load this data in your MATLAB, then download this data from the following link: mandrill.mat. Then, run the above load command again. Display this matrix on your screen by:

```
>> image(X); colormap(map)
```

Submit this figure as part of your HW.

(b) Compute the SVD of this mandrill image and plot the distribution of its singular values on your screen. Note that the MATLAB `svd` function returns three matrices U, S, V for a given input matrix. So, the singular values are nicely plotted by:

```
>> stem(diag(S)); grid
```

Submit this image as part of your HW.

(c) Let σ_j, $\boldsymbol{u}_j$, $\boldsymbol{v}_j$ be the jth singular value, the jth left and right singular vectors of the mandrill image, respectively. In other words, they are `S(j,j)`, `U(:,j)`, `V(:,j)` of the SVD of X in MATLAB. Let us define the rank k approximation of the image X as

$$X_k := \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^\top + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^\top .$$

Then, for $k = 1, 6, 11, 31$, compute X_k of the mandrill, and display the results. Fit these four images in one page by using subplot function in MATLAB (i.e., use `subplot(2,2,1)` to display the first image, `subplot(2,2,2)` to display the second image, etc.)

(d) For $k = 1, 6, 11, 31$, display the residuals, i.e., $X - X_k$, fit them in one page and **submit these as part of your HW.**

(e) For $k = 1, 6, 11, 31$, compute $\|X - X_k\|_2$ by the `norm` function of MATLAB. Then, compare the results with σ_{k+1}. More precisely, compute the relative error and report the results:

$$\frac{|\sigma_{k+1} - \|X - X_k\|_2\,|}{\sigma_{k+1}}.$$

Exercise 4

Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine the SVD of A by hand.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit circle in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Use the SVD to find A^{-1}. (You must show your work and how you used SVD.)