MAT 108
Exam 1

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): Solutions

Name (print): Solutions

Problem	Points Possible	Points Earned
Q1	15	
Q1 style	5	
Q2	15	
Q2 style	5	
Q3	15	
Q3 style	5	
Total:	60	

- You have 45 minutes to complete this exam. If you are done early, you may leave after handing me your exam packet and any scratch paper you used.
- The total number of points assigned to each problem part is shown here and within the exam.
- **You will be graded on both mathematical reasoning and on writing style.** In particular, your proofs must be in full sentences and neat, emulating the style of a typeset mathematical document.
- It is highly recommended that you use scratch paper to figure out how you want to structure your proof, and then write down your final answer in the test packet.
- You will be provided with scratch paper, and can obtain more from me during the exam. Write your name at the top of each scratch sheet. At the end of the exam, hand in both your exam packet and your scratch paper. **You will only be graded on the written proofs in your exam packet.**

1. (15 points + 5 style points) Using the axioms of the integers stated below, prove the proposition.

Axioms of the integers Let $m, n, p \in \mathbb{Z}$.

(a) $n + m = m + n$ (commutativity of addition)

(b) $(m + n) + p = m + (n + p)$ (associativity of addition)

(c) $m \cdot (n + p) = m \cdot n + m \cdot p$ (distributivity)

(d) $m \cdot n = n \cdot m$ (commutativity of multiplication)

(e) $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ (associativity of multiplication)

(f) There exists an integer 0 such that, whenever $m \in \mathbb{Z}$, $m + 0 = m$. (identity element for addition)

(g) There exists an integer 1 such that $1 \neq 0$ and whenever $m \in \mathbb{Z}$, $m \cdot 1 = m$ (identity element for multiplication)

(h) $\forall\, m \in \mathbb{Z}$, $\exists\, (-m) \in \mathbb{Z}$ such that $m + (-m) = 0$ (additive inverse)

(i) Let $m, n, p \in \mathbb{Z}$. If $m \neq 0$ and $m \cdot n = m \cdot p$, then $n = p$. (cancellation)

Proposition. Given $k, n \in \mathbb{Z}$ such that $k, n \neq 0$ and $k \mid n$, show that there *exists* a *unique* $p \in \mathbb{Z}$ such that $k \cdot p = n$.

We first show existence. Since $k|n$, by definition, there exists $p \in \mathbb{Z}$ such that $k \cdot p = n$.

Now we consider uniqueness. Suppose we have $p_1, p_2 \in \mathbb{Z}$ such that $k \cdot p_1 = n$ and $k \cdot p_2 = n$.
Then $k \cdot p_1 = k \cdot p_2$.
Since $k \neq 0$, the cancellation axiom tells us that $p_1 = p_2$.
So p is unique.

$\square$

2. (15 points + 5 style points) Prove that for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

We show this by inducting on n.

In the base case, $n=1$. Then $n^3 - n = 1^3 - 1 = 0 = 0 \cdot 3$, so $3 \mid n^3 - n$.

Now assume that for some $n \geq 1$, $3 \mid n^3 - n$.

We will show that $3 \mid (n+1)^3 - (n+1)$.

Expanding, we have

$$\begin{aligned} &(n+1)^3 - (n+1) \\ &= n^3 + 3n^2 + 3n + 1 - n - 1 \end{aligned}$$

which can be rearranged as

$$\begin{aligned} &= (n^3 - n) + 3n^2 + 3n \\ &= (n^3 - n) + 3(n^2 + n). \end{aligned}$$

By the induction hypothesis, there exists $p \in \mathbb{Z}$ such that $n^3 - n = 3 \cdot p$. Therefore the above expression can be rewritten as

$$\begin{aligned} &= 3 \cdot p + 3(n^2 + n) \\ &= 3(p + n^2 + n). \end{aligned}$$

Therefore $(n+1)^3 - (n-1) = 3(p + n^2 + n)$ where $p + n^2 + n \in \mathbb{Z}$, so $3 \mid (n+1)^3 - (n+1)$.

$\square$

3. (15 points + 5 style points) Let R, S, and T be sets. Show that

$$R \cap (S - T) = (R \cap S) - (R \cap T).$$

Hint: Recall that for two sets A and B, the set difference $A - B$ is defined to be $\{x : x \in A \text{ and } x \notin B\}$.

To show this set equivalence, we show double inclusion.

First, we show that $R \cap (S-T) \subseteq (R \cap S) - (R \cap T)$.

Let $x \in R \cap (S-T)$. Then $x \in R$ and $x \in S$ but $x \notin T$.

Since $x \in R$ and $x \in S$, we have $x \in R \cap S$.

Since $x \notin T$, x is also not in $R \cap T$, which contains T.

We have shown that $x \in R \cap S$ and $x \notin R \cap T$, so

$$x \in (R \cap S) - (R \cap T).$$

Next, we show that $(R \cap S) - (R \cap T) \subseteq R \cap (S-T)$.

Let $x \in (R \cap S) - (R \cap T)$. Then $x \in R \cap S$ but $x \notin R \cap T$.

We want to show that $x \in R$ and $x \in S - T$.

Since $x \in R \cap S$, $x \in R$ and $x \in S$.

It remains to show that $x \notin T$. Since $x \notin R \cap T$, we have $x \notin R$ or $x \notin T$. But we know that $x \in R$, so it must be that $x \notin T$.

We have $x \in R$ and $x \in S$ and $x \notin T$, so $x \in R \cap (S-T)$.

We have shown that $R \cap (S-T)$ and $(R \cap S) - (R \cap T)$ include each other, so they are equal.