MAT 108

Exam 2

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): Solutions   Name (print): Solutions

Problem	Points Possible	Points Earned
Q1	15	
Q1 style	5	
Q2	15	
Q2 style	5	
Q3	15	
Q3 style	5	
Total:	60	

- You have 45 minutes to complete this exam. If you are done early, you may leave after handing me your exam packet and any scratch paper you used.
- The total number of points assigned to each problem part is shown here and within the exam.
- **You will be graded on both mathematical reasoning and on writing style.** In particular, your proofs must be in full sentences and neat, emulating the style of a typeset mathematical document.
- It is highly recommended that you use scratch paper to figure out how you want to structure your proof, and then write down your final answer in the test packet.
- You will be provided with scratch paper, and can obtain more from me during the exam. Write your name at the top of each scratch sheet. At the end of the exam, hand in both your exam packet and your scratch paper. **You will only be graded on the written proofs in your exam packet.**

1. (15 points + 5 style points) Consider the following relation on $\mathbb{R}$:

$$x \sim y \qquad \text{iff} \qquad x - y \in \mathbb{Z}.$$

Prove that $\sim$ is a well-defined *equivalence relation*.

(Recall that an equivalence relation must satisfy *reflexivity, symmetry, and transitivity.*)

<u>Reflexivity</u> Since $x - x = 0 \in \mathbb{Z}$, $x \sim x$.

<u>Symmetry</u> If $x \sim y$, then $x - y = n \in \mathbb{Z}$.
Then $y - x = -n \in \mathbb{Z}$ as well
So $y \sim x$.

<u>Transitivity</u> Suppose $x \sim y$ and $y \sim z$. Then
$x - y = n \in \mathbb{Z}$ and $y - z = m \in \mathbb{Z}$.
Then $x - z = x - y + y - z = n + m \in \mathbb{Z}$
So $x \sim z$ as well.

2. (15 points + 5 style points) Consider the sequence of real numbers $(x_k)_{k=1}^{\infty}$ given by

$$x_k = \frac{(-1)^k}{k}.$$

This is the sequence which begins with the terms

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \cdots.$$

(a) Prove that $\lim_{k\to\infty} x_k = 0$. *(You must prove your answer using the ε-N definition of limits!)*

Let $\varepsilon > 0$. Since $\mathbb{N}$ is unbounded,

there exists $N \in \mathbb{N}$ such that

$N > \frac{1}{\varepsilon}$, and so $\frac{1}{N} < \varepsilon$.

Then for all $n \geq N$,

$$\left| \frac{(-1)^n}{n} - 0 \right| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So $\lim\limits_{k\to\infty} \frac{(-1)^k}{k} = 0$.

(b) Let A be the set $\{x_k\}_{k\in\mathbb{N}}$. What is $\sup A$? *(Prove your answer!)*

We claim that $\sup A = \frac{1}{2}$.

First of all, $-1, \frac{1}{2} \leq \frac{1}{2}$.

For the terms $\frac{(-1)^k}{k}$ for $k \geq 3$, we have

$$\left| \frac{(-1)^k}{k} \right| = \frac{1}{k}.$$

Since $k \geq 3$, $\frac{1}{k} \leq \frac{1}{3} < \frac{1}{2}$. So

$$\left| \frac{(-1)^k}{k} \right| < \frac{1}{2},$$ and in particular,

$$\frac{(-1)^k}{k} < \frac{1}{2}.$$

Therefore for all k, $\frac{(-1)}{k} \leq \frac{1}{2}$.

For any other $u < \frac{1}{2}$, u is not an upper bound since $u <$ the 2nd term of the sequence.

Therefore $\frac{1}{2}$ is the <u>least</u> upper bound.

(You could also prove that $\frac{1}{2} = \max A$ and then conclude $\sup A = \max A = 1/2$.)

3. (15 points + 5 style points) Consider the set

$$(\mathbb{Z}_2)^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Define a norm on $(\mathbb{Z}_2)^4$ by

$$\|xyzw\| = \|(x, y, z, w)\| = 8x + 4y + 2z + w,$$

where addition and multiplication on the right-hand side occurs in $\mathbb{Z}$, with $x, y, z, w \in \{0, 1\}$. This defines a distance function $d : (\mathbb{Z}_2)^4 \to \mathbb{R}_{\geq 0}$. For example, the distance between 1101 and 0111 is

$$\begin{aligned} d(1101, 0111) &= \|(1, 1, 0, 1) - (0, 1, 1, 1)\| \\ &= \|(1 - 0, 1 - 1, 0 - 1, 1 - 1)\| \\ &= \|(1, 0, 1, 0)\| \\ &= 8 + 0 + 2 + 0 = 10. \end{aligned}$$

(a) Compute the distance between 0111 and 0000.

The distance between 0111 and 0000 is

$$d(0111, 0000) = 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 = 7$$

(b) What is the maximum value of d? In other words, if $A = \{d(p,q) \ : \ p,q \in (\mathbb{Z}_2)^4\}$, what is $\max A$? Justify (i.e. prove) your answer.

We will show the maximum value of d is

$$1 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 = 15.$$

Given any $p, q \in (\mathbb{Z}_2)^4$, if all the coordinates of p and q differ, then $d(p,q) = 15$ and is calculated by the equation above

If p and q agree in any coordinate, then some of the coefficients of the powers of 2 in the equation would be replaced by 0s. Since $0 < 1$, the resulting sum would be (strictly) less than 15.

Therefore $\forall\ p, q \in (\mathbb{Z}_2)^4 \leq 15$ and

$d(0000, 1111) = 15$, so $\max A = 15$.

See next page for another valid proof that looks quite different.

Another valid proof, for reference:

We will show that the maximum value of d is

$$1\cdot 8 + 1\cdot 4 + 1\cdot 2 + 1\cdot 1 = 15.$$

First, note that $d(p,q) = \| p-q \|$, so $\max A$ is equal

$\max \{ \| xyzw \| : xyzw \in (\mathbb{Z}_2)^4 \}$, where $x,y,z,w \in \{0,1\}$.

The function $\| xyzw \| = 8x + 4y + 2z + w$ is maximized

when the values of the real numbers x,y,z,w are

maximized, i.e. when $x=y=z=w=1$.

Therefore $\max A = \| 1111 \| = 15$.