MAT 108: Mock Final Solutions

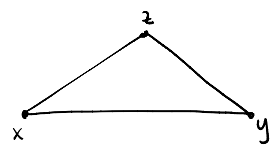
1. Assume the following two *axioms*:

   A1 The area of a planar rectangle with sides $a, b \in \mathbb{R}$ is the product $a \cdot b$.

   A2 The area of two planar figures which intersect at most along edges is the sum of areas of each of the planar figures.

   Use Axioms 1 and 2 to deduce that the area of the triangle with height $h \in \mathbb{R}$ and base $b \in \mathbb{R}$ equals $(b \cdot h)/2$.

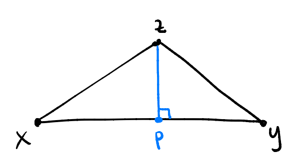
Let T be a planar triangle. Without loss of generality, we may position T so that the edge opposite the largest angle is horizontal. Then, label the vertices as in the figure below:



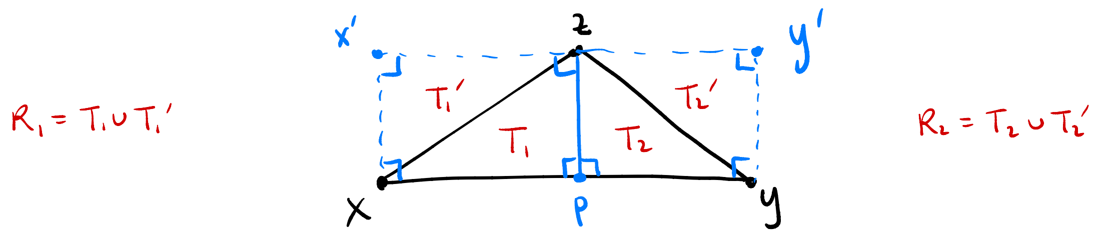
$x = (x_1, x_2)$
$y = (y_1, y_2)$
$z = (z_1, z_2)$

Since the largest angle is at z, the angles at x and y must be acute; otherwise, if one of x, y is obtuse (and thus z is obtuse as well), then the sum of the angles would be greater than $90° + 90° = 180°$.

Then the vertical line through z will intersect the edge $\overline{xy}$ at some point p:



Define the points $x' = (x_0, z_1)$ and $y' = (y_0, z_1)$:



Now observe that the following pairs of triangles have the same shape and size, and thus area as well:

① T_1 and T_1' ② T_2 and T_2'.

(This is because T_i and T_i' are similar right triangles that share their longest side, the hypotenuse.)

Let $A(\cdot)$ denote the area function on closed shapes.

By Axiom 2, $A(R_1) = A(T_1) + A(T_1') = 2 \cdot A(T_1)$

and $A(R_2) = A(T_2) + A(T_2') = 2 \cdot A(T_1)$.

The base of T is $\overline{xy}$, and has length b; the height of T is h, the length of $\overline{pz}$.

The rectangle $R_1 \cup R_2$ also has base length b and height h.

By Axiom 1, $A(R_1 \cup R_2) = b \cdot h$.

By Axiom 2, $A(R_1 \cup R_2) = A(R_1) + A(R_2)$.

Therefore $b \cdot h = A(R_1 \cup R_2) = A(R_1) + A(R_2) = 2A(T_1) + 2A(T_2)$,

Since $T = T_1 \cup T_2$ (intersecting only along $\overline{pz}$), by Axiom 2,

$$A(T) = A(T_1) + A(T_2) = \frac{1}{2} A(R_1 \cup R_2) = \frac{1}{2} bh.$$

∎

2. Prove that for $n \geq 8$,

$$3n^2 + 3n + 1 < 2^n.$$

We will prove the statement by induction.

(Base case) In the base case, $n=8$. We verify that

$$3n^2 + 3n + 1 = 3 \cdot 64 + 3 \cdot 8 + 1 = 217 < 256 = 2^8.$$

(Induction Step) Now assume that for some $n \geq 8$,

$$3n^2 + 3n + 1 < 2^n.$$

We will show that $3(n+1)^2 + 3(n+1) + 1 < 2^{n+1}$.

Expand the left-hand side:

$$\begin{aligned}
&3(n+1)^2 + 3(n+1) + 1 \\
&= 3(n^2 + 2n + 1) + 3n + 3 + 1 \\
&= 3n^2 + 6n + 3 + 3n + 3 + 1 \\
&= (3n^2 + 3n + 1) + (6n + 6).
\end{aligned}$$

By the induction hypothesis, $3n^2 + 3n + 1 < 2^n$. Since $2^{n+1} = 2^n + 2^n$, it remains to show that $6n+6 < 2^n$, or equivalently, $3n+3 < 2^{n-1}$, for all $n \geq 8$.

<u>Claim</u> For $n \geq 8$, $6n+6 < 2^n$.

Pf. In the base case, $n=8$:

$$6 \cdot 8 + 6 = 54 < 256 = 2^8.$$

Assume that for some $n \geq 8$, $6n+6 < 2^n$.

Then $6(n+1) + 6 = (6n+6) + 6$.

By the induction hypothesis,

$$6(n+1) + 6 = (6n+6) + 6 < 2^n + 6.$$

Since $n \geq 8$, $6 < 2^n$, so

$$2^n + 6 < 2^n + 2^n = 2^{n+1}.$$ ■

In summary,

$$3(n+1)^2 + 3(n+1) + 1 = (3n^2+3n+1) + (6n+6) < 2^n + 2^n = 2^{n+1}$$

by the induction hypothesis and Claim. ■

3. Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

Suppose, by way of contradiction, that there exist $a, b \in \mathbb{N}$ such that $a^2 - b^2 = 10$.

Then $a^2 = b^2 + 10$, so $b^2 + 10$ is a perfect square.

Since $b^2 + 10 > b^2$, there must be some $c \in \mathbb{N}$ such that

$$(b+c)^2 = b^2 + 10.$$

Expanding the LHS, we have

$$(b+c)^2 = b^2 + 2bc + c^2 = b^2 + 10.$$

Therefore $2bc + c^2 = 10$.

Since $2bc \in \mathbb{N}$, $c^2 < 2bc + c^2 = 10$, so $c \in \{1, 2, 3\}$.

But since $2bc$ and 10 are both even, $c^2 = 10 - 2bc$ must also be even, and so $c = 2$.

Then $2bc + c^2 = 2b \cdot 2 + 2^2 = 4b + 4 = 10$, so $4b = 6$.

Then $b = 6/4 \notin \mathbb{N}$, so we have a contradiction.

4. Consider the following recursively defined sequence:

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + 1 \qquad \forall\, n \in \mathbb{N}.$$

(a) Show that (x_n) is increasing and bounded above.

(b) Prove that (x_n) converges and find its limit.

(a) We need to show that for all $n \geq 1$,

$$x_{n+1} = \frac{x_n}{2} + 1 > x_n, \text{ ie that } \frac{x_n}{2} + 1 - x_n > 0.$$

Since $\frac{x_n}{2} + 1 - x_n = 1 - \frac{x_n}{2}$, it suffices to show that $\forall\, n \geq 1$, $x_n < 2$; we show this by induction:

In the base case, $n=1$, and indeed $x_1 = 1 < 2$.

Suppose that for some n, $x_n < 2$. Then

$$x_{n+1} = \frac{x_n}{2} + 1 < 1 + 1 < 2.$$

Therefore (x_n) is increasing, and bounded above by 2.

(b) The sequence is also bounded below by $x_1 = 1$, since it's increasing. So (x_n) is bounded and monotone, so by the Monotone Convergence Theorem, (x_n) converges.

To find the actual limit, we will find an explicit formula for x_n.

<u>Claim</u> $x_n = \frac{2^n - 1}{2^{n-1}}$

pf. In the base case, $n=1$, and indeed $x_1 = 1 = \frac{2^1 - 1}{2^0}$.

Assume that for some $n \geq 1$, $x_n = \frac{2^n - 1}{2^{n-1}}$.

Then $$x_{n+1} = \frac{x_n}{2} + 1 = \frac{2^n - 1}{2^n} + 1 = \frac{2^n - 1 + 2^n}{2^n} = \frac{2^{n+1} - 1}{2^{(n+1)-1}}.$$ ∎

We now show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2^n-1}{2^{n-1}} = 2$.

Observe that $\frac{2^n-1}{2^{n-1}} = \frac{2^n}{2^{n-1}} - \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{N-1} > \frac{1}{\varepsilon}$.

Then for all $n \geq N$,

$$|2 - x_n| = \left|2 - \left(2 - \frac{1}{2^{n-1}}\right)\right| = \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \varepsilon.$$

5. Consider the set of black and white colorings of the numbers in the interval $[0, 1] \subset \mathbb{R}$:

$$C := \{c : [0, 1] \to \{\text{black}, \text{white}\}\}.$$

Show that $\operatorname{card} C > \operatorname{card} \mathbb{R}$. *Note the strict inequality.*

We first define a bijection between C and $\mathcal{P}([0,1])$.

Define $f: C \longrightarrow \mathcal{P}([0,1])$ by $c \mapsto X_c$,

where $X_c = \{ p \in [0,1] : c(p) = \text{black} \}$.

We can also define an inverse: let $g: \mathcal{P}([0,1]) \longrightarrow C$ be defined by $X \subset [0,1] \mapsto c_X$, where for $p \in [0,1]$,

$c_X(p) = \text{black}$ iff $p \in X$.

Since f has an inverse, f is a bijection, so

$$\text{card}(C) = \text{card}(\mathcal{P}([0,1])).$$

Since $(0,1) \subset [0,1]$, $\text{card}((0,1)) \leq \text{card}([0,1])$.

By PS 9, $\text{card}(\mathbb{R}) = \text{card}((-\frac{\pi}{2}, \frac{\pi}{2})) = \text{card}((0,1))$.

Thus $\text{card}(\mathbb{R}) \leq \text{card}([0,1]) < \text{card}\, \mathcal{P}([0,1])$,

where the strict inequality come from the theorem that for any set A, $\text{card}\, A < \text{card}\, \mathcal{P}(A)$.