MAT 108: Problem Set 1

Solutions

Due 1/17/23 at 11:59 pm on Canvas

Exercise 1

Prove Proposition 1.14:

Proposition. For all $m \in \mathbb{Z}$, $m \cdot 0 = 0 = 0 \cdot m$.

Reminder. *To prove an equation holds, we start at one end and use a chain of known equalities to arrive at the other end.*

SOLUTION.
Let $m \in \mathbb{Z}$. By commutativity of multiplication, we have

$$m \cdot 0 = 0 \cdot m.$$

We will now show that $m \cdot 0 = 0$. Firstly, by additive identity, we know

$$m \cdot 0 = m \cdot (0 + 0).$$

Using distributivity,

$$m \cdot (0 + 0) = m \cdot 0 + m \cdot 0.$$

That is, we now know have

$$m \cdot 0 = m \cdot 0 + m \cdot 0.$$

Finally, we add $-(m \cdot 0)$ to both sides

$$m \cdot 0 + (-(m \cdot 0)) = m$$

Exercise 2

a) Prove Proposition 1.24:

Proposition. Let $x \in \mathbb{Z}$. If $x \cdot x = x$, then $x = 0$ or $x = 1$.

b) Prove Proposition 1.26:

Proposition. Let $m, n \in \mathbb{Z}$. If $m \cdot n = 0$, then $m = 0$ or $n = 0$.

SOLUTION.

a) If $x = 0$, then we are done. (Note that indeed, $0 \cdot 0 = 0$.)

Now consider the case where $x \neq 0$. In this case, we can use the cancellation axiom on the equation

$$x \cdot x = x$$

and arrive at $x = 1$.

b) If $m = 0$, then we are done. So assume that $m \neq 0$. In this case, we have

$$m \cdot n = 0 = m \cdot 0.$$

We can apply the cancellation axiom to the equation

$$m \cdot n = m \cdot 0$$

to see that $n = 0$.

Exercise 3

In this exercise, we first define divisibility using only the multiplication operation $\cdot$ on $\mathbb{Z}$:

Definition. Let $m, n \in \mathbb{Z}$. If there exists a $q \in \mathbb{Z}$ such that $m = n \cdot q$, then we say m *is divisible by* n, or equivalently, n *divides* m. We denote this relationship by $n \mid m$.

Prove the following statements.

a) 0 is divisible by every integer.

b) If m is an integer not equal to 0, then m is not divisible by 0.

c) Let $x \in \mathbb{Z}$. If x has the property that for all $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.

Hint. *In higher math courses, it is often beneficial to read the book sections in addition to going to lecture. In particular, both lecture and the book are pointedly useful for part (c) of the above exercise.*

Solution.

a) Let $n \in \mathbb{Z}$ be any integer. To show that n divides 0, by definition, we need to show that there exists a $q \in \mathbb{Z}$ such that

$$0 = n \cdot q.$$

Setting $q = 0$, we get $n \cdot q = n \cdot 0 = 0$, so $n \mid 0$ indeed.

b) Let $m \neq 0$ be a nonzero integer. For all $q \in \mathbb{Z}$, $0 \cdot q = 0$, so $m \neq 0 \cdot q$. That is, for all $q \in \mathbb{Z}$, q cannot divide m.

c) Suppose $x \in \mathbb{Z}$ satisfies the property that for all $m \in \mathbb{Z}$, $mx = m$. In particular, $mx = m$ holds for all $m \neq 0$. In such cases (for example, when $m = 17$), we can use the cancellation axiom on the left-most and right-most sides of the equation

$$mx = m = m \cdot 1$$

to deduce that $x = 1$.