MAT 108: Problem Set 2

(ADD NAME)

Due 1/24/23 at 11:59 pm on Canvas

Reminders:

- Put your name at the top!
- You will receive feedback on PS1 by next Tuesday, 1/24. PS1 revisions are due Friday, 1/27 at 11:59 pm.
- Dr. Zhang will be away all of next week (no instructor office hours). I will be in Germany, where the time is 9 hours ahead of California. You may email me your questions, and I will respond once daily as usual. The TA will still hold his office hours.

Another reminder Figuring out how to prove something should feel like doing a puzzle. Writing down and expressing your proof should feel like you're trying to write in a new language. Trust the process!

How much detail is needed? In PS2, you no longer need to cite the axioms / propositions from Chapter 1. For example, it's clear to the audience (e.g. your classmates) that $-0 = 0$.

On the other hand, in Chapter 2, we defined the natural numbers $\mathbb{N}$ using a set of axioms that are not very obvious to your peers. You should cite these axioms as you use them.

Exercise 1

Prove that $1 \in \mathbb{N}$ via a proof by contradiction. Then, deduce that that if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Remark. The phrase *deduce that* here is indicating that the second statement follows quite immediately from the first.

SOLUTION.

Exercise 2

Definition. Let $m, n \in \mathbb{Z}$. If $m - n \in \mathbb{N}$, then we say n *is less than* m, and write $n < m$. We also say m *is greater than* n, and write $m > n$.

Prove that there exists no integer x such that $0 < x < 1$.

SOLUTION.

Exercise 3

Use induction to prove that for any $n \in \mathbb{N}$, the following formula holds:

$$1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}.$$

Exercise 4

Definition. Let $p \in \mathbb{N}$. If the only $k \in \mathbb{N}$ such that $k \mid p$ are $k = 1$ and $k = p$, then p is *prime*.

Prove that there are infinitely many prime numbers.

Remark. We haven't rigorously discussed the term *infinite* just yet. We will discuss cardinality in detail in a few weeks. For now, *infinite* means *not finite*.

You should begin by assuming there are finitely many prime numbers, so that you can label them $p_1, p_2, p_3, \ldots, p_n$ for some finite $n \in \mathbb{N}$. Then try to derive a contradiction.

You may use the following Lemma without proof:

Lemma. Let p be a prime number, and let $m \in \mathbb{N}$. If p divides m, then p does not divide $m + 1$.

(Proving this lemma will be easier once we've developed the language of *modular arithmetic* later on in the course.)

SOLUTION.