MAT 108: Problem Set 3

Solutions

Due 1/31/23 at 11:59 pm on Canvas

Reminders:

- Your homework submission must be typed up in full sentences, with proper mathematical formatting.
- Midterm Exam 1 is Wednesday, 2/1 next week. This problem set contains both regular "graded" exercises as well as review problems for the exam. The review problems will only be graded once for completion.
- You will receive feedback on PS2 by next Tuesday, 1/31. PS2 revisions are due Friday, 2/3 at 11:59 pm.

Another reminder As with many math classes, success comes with practice. To study for the upcoming exam, I would recommend trying to prove the propositions in the textbook (only some of them were on your homework) rather than memorizing facts.

Exercise 1

(Graded, 10 points) Prove that $(A \cup B)^c = A^c \cap B^c$. (This is part (b) of Theorem 5.15 (De Morgan's Laws).)

Solution.

In order to prove this set equivalence, we will show double inclusion.

First, we show that $(A \cup B)^c \subseteq A^c \cap B^c$. Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, i.e. $x \notin A$ or $x \notin B$. Without loss of generality, we may assume that $x \notin A$. Then $x \in A^c \subset A^c \cup B^c$.

Next, we show that $(A^c \cap B^c) \subseteq (A \cup B)^c$. In other words, we want to show that if $x \in (A^c \cap B^c)$, then x is not in $A \cup B$. We will instead show the contrapositive, which states that if $x \in A \cup B$, then x is not in $(A^c \cap B^c)$.

Suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$, so without loss of generality we may assume $x \in A$. Then $x \notin A^c$, so it is in particular not in $A^c \cap B^c$, which is a subset of A^c.

□

Exercise 2

(Graded, 15 points) Let $A_1, A_2, A_3, \ldots$ be a sequence of sets.

(a) Develop recursive definitions for

$$\bigcup_{j=1}^{k} A_j \qquad \text{and} \qquad \bigcap_{j=1}^{k} A_j.$$

(b) Write down and prove an analogue for Theorem 5.15 (a) for these unions and intersections.

(c) Write down and prove an analogue for Theorem 5.15 (b) for these unions and intersections.

SOLUTION.

(a) Explicitly, $\bigcup_{j=1}^{k} A_j$ should mean $A_1 \cup A_2 \cup \cdots \cup A_k$. To define this recursively, we let

$$\bigcup_{j=1}^{k} A_j = A_1$$

and use the recurrence relation

$$\bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}.$$

Similarly, to define $\bigcap_{j=1}^{k} A_j$, we let

$$\bigcap_{j=1}^{k} A_j = A_1$$

and use the recurrence relation

$$\bigcap_{j=1}^{k+1} A_j = \left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}.$$

(b) Let $A_1, \ldots, A_k$ be sets. We will show that

$$\left(\bigcap_{j=1}^{k} A_j\right)^c = \bigcup_{j=1}^{k} (A_j^c).$$

Recall that Theorem 5.15(a) tells us that

$$(A \cap B)^c = A^c \cup B^c.$$

In the base case, $k = 2$, and the equation is equivalent to Theorem 5.15 (a), by setting $A = A_1$ and $B = A_2$.

Now assume that the equality holds for some $k \geq 2$. Let $A = \bigcap_{j=1}^{k} A_j$ and $B = A_{k+1}$. Theorem 5.15(a) tells us that

$$\left(\bigcap_{j=1}^{k+1} A_j\right)^c = \left(\bigcap_{j=1}^{k} A_j\right)^c \cup A_{k+1}^c.$$

By the induction hypothesis, $\left(\bigcap_{j=1}^{k} A_j\right)^c = \bigcup_{j=1}^{k}(A_j^c)$.

Therefore
$$\left(\bigcap_{j=1}^{k+1} A_j\right)^c = \left(\bigcup_{j=1}^{k} A_j^c\right) \cup A_{k+1}^c = \bigcup_{j=1}^{k+1}(A_j)^c.$$

□

(c) *Solution omitted. This is very similar to the previous proof.*

Exercise 3

(Graded, 10 points) Prove that if $a^2(b^2 - 2b)$ is odd, then a and b are (both) odd. (*Hint:* Try proving the contrapositive of the statement instead.)

SOLUTION.

We will instead prove the contrapositive. That is, we will show that if at least one of a or b is even, then $a^2(b^2 - 2b)$ will be even. Since a and b are not interchangeable in the statement, we need to consider both cases separately.

First, consider the case where a is even, so that there is some $k \in \mathbb{Z}$ such that $a = 2k$. Then

$$a^2(b^2 - 2b) = 2k \cdot a(b^2 - 2b) = 2(ka(b^2 - 2b)$$

is even.

Second, consider the case where b is even, so that there is some $j \in \mathbb{Z}$ where $b = 2j$. Then

$$a^2(b^2 - 2b) = b \cdot a^2(b - 2) = 2j \cdot a^2(b - 2) = 2(ja^2(b - 2))$$

is even. □

Exercise 4

(Graded, 10 points) The Fibonacci numbers $(f_j)_{j=1}^{\infty}$ are defined by $f_1 := 1$, $f_2 := 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 3.$$

Prove that for all $k, m \in \mathbb{N}$ $(m \geq 2)$,

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}.$$

(*Hint:* First decide the statements your induction argument intends to prove. What variable are you inducting on? Then, remember that the recurrence relation for Fibonacci numbers is an available and relevant tool. You may wish to take a look at the paragraph about *strong induction* in the textbook.)

SOLUTION.

(There are many similar ways to prove this statement. Here is one such way.)

Let n be a natural number. Let $P(n)$ be the statement the following statement:

For all $m, k \in \mathbb{N}$ such that $m \geq 2$ and $k \geq 1$ (so that $m-1 \in \mathbb{N}$) and $n = m+k$,

$$f_{m+k} = f_{m-1}f_k + f_m + f_{k+1}.$$

We will induct on n.

In the base case, $m = 2$ and $k = 1$, so we consider $n = 3$. Indeed,

$$f_{m-1}f_k + f_m + f_{k+1} = f_1 \cdot f_1 + f_2 \cdot f_2 = 1 \cdot 1 + 1 \cdot 1 = 2 = f_3.$$

Now consider a fixed $n > 3$, and assume that $P(j)$ holds for all $3 \leq j \leq n$. We want to show that $P(n+1)$ holds. Observe that it suffices to check that for all m, k such that $m + k = n$, the following two equations hold:

$$f_{(m+1)+k} = f_m f_k + f_{m+1} f_{k+1} \tag{1}$$

and

$$f_{m+(k+1)} = f_{m-1} f_{k+1} + f_m f_{k+2}. \tag{2}$$

To prove Equation 1, we begin from the right-hand side of the equation and use the recurrence relation:

$$\begin{aligned} f_m f_k + f_{m+1} f_{k+1} &= (f_{m-1} + f_{m-2}) f_k + (f_m + f_{m-1}) f_{k+1} \\ &= (f_{m-1} f_k + f_{m-2} f_k) + (f_m f_{k+1} + f_{m-1} f_{k+1}). \end{aligned}$$

After reordering the terms, this is

$$= (f_{m-1} f_k + f_m f_{k+1}) + (f_{m-2} f_k + f_{m-1} f_{k+1}).$$

By the induction hypothesis, $P(n)$ is true, so

$$(f_{m-1} f_k + f_m f_{k+1}) = f_{m+k} = f_n,$$

and $P(n-1)$ is true, so

$$(f_{m-2} f_k + f_{m-1} f_{k+1}) = f_{(m-1)+k} = f_{n-1}.$$

Therefore

$$f_m f_k + f_{m+1} f_{k+1} = f_n + f_{n-1} = f_{n+1} = f_{m+k+1}.$$

This is Equation 1.

To prove Equation 2, we again begin on the right-hand side. We similarly expand the two terms using the recurrence relation:

$$\begin{aligned} & f_{m-1} f_{k+1} + f_m f_{k+2} \\ & = f_{m-1}(f_k + f_{k-1}) + f_m (f_{k+1} + f_k) \\ & = (f_{m-1} f_k + f_{m-1} f_{k-1}) + (f_m f_{k+1} + f_m f_k) \end{aligned}$$

and then combine terms using the induction hypothesis:

$$\begin{aligned} & = f_{m-1} f_k + f_m f_{k-1} + f_{m-1} f_{k-1} + f_m f_{k+1} \\ & = (f_m f_{k-1} + f_{m-1} f_{k-1}) + (f_{m-1} f_k + f_m f_{k+1}) \\ & = f_{m+k} + f_{m+(k-1)}. \end{aligned}$$

This proves Equation 2.

Exercise 5

(Graded once for completion, 10 points) Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. Prove that if $\prod_{i=1}^{n} a_i = 0$, then for some i, $a_i = 0$.

SOLUTION.

Let $P(k)$ be the statement if $\prod_{i=1}^{k} a_i = 0$, then for some i such that $1 \leq i \leq k$, $a_i = 0$.

On PS2, we showed that if $m, m' \in \mathbb{Z}$ and $m \cdot m' = 0$, then $m = 0$ or $m' = 0$. Setting $m = a_1$ and $m' = a_2$, we see that $P(2)$ is true.

Now suppose $P(k)$ is true, for $2 \leq k < n$. We want to show that $P(k+1)$ is true. There are two cases to consider.

First, consider the case where $\prod_{i=1}^{k} a_i = 0$. Then by the induction hypothesis, some $a_i = 0$ such that $1 \leq i \leq k$. In particular, $i \leq k+1$, so $P(k+1)$ is true.

In other other case, $\prod_{i=1}^{k} a_i \neq 0$. Then setting $m = \prod_{i=1}^{k} a_i$ and $m' = a_{i+1}$, we know by PS2 that $m' = a_{k+1} = 0$, since at least one of m and m' must be zero.

In either case, we have exhibited an a_i (where $i \leq k+1$ that is zero. By induction, $P(n)$ is true. □

Exercise 6

(Graded once for completion, 5 points) Show that for all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.

SOLUTION.

We will prove this by induction on k.

In the base case, $k = 1$, and

$$k^4 - 6k^3 + 11k^2 - 6k = 1 - 6 + 11 - 6 = 0$$

which is indeed divisible by 4 (as $0 = 0 \cdot 4$).

Now assume that $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4, i.e. there is some $a \in \mathbb{Z}$ such that $k^4 - 6k^3 + 11k^2 - 6k = 4a$. We will show that 4 divides

$$(k+1)^4 - 6(k+1)^3 + 11(k+1)^2 - 6(k+1).$$

Expanding this (perhaps using the binomial theorem!) and combining like terms, we get

$$\begin{aligned}
&(k^4 + 4k^3 + 6k^2 + 4k + 1) - 6(k^3 + 3k^2 + 3k + 1) + 11(k^2 + 2k + 1) - 6(k+1) \\
&= k^4 + (4-6)k^3 + (6 - 18 + 11)k^2 + (4 - 18 + 22 - 6)k + (1 - 6 + 11 - 6) \\
&= (k^4 - 6k^3 + 11k^2 - 6k) + (4k^3 - 12k^2 + 8k + 0) \\
&= 4a + 4(k^3 - 3k^2 + 2k) \\
&= 4(a + k^3 - 3k^2 + 2k).
\end{aligned}$$

where the second to last equality follows from the induction hypothesis. Since

$$(k+1)^4 - 6(k+1)^3 + 11(k+1)^2 - 6(k+1) = 4(a + k^3 - 3k^2 + 2k),$$

we have shown $(k+1)^4 - 6(k+1)^3 + 11(k+1)^2 - 6(k+1)$ is divisible by 4.

□