# MAT 108: Problem Set 4

Solutions

Due 2/7/23 at 11:59 pm on Canvas

**Reminders:**

- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS3 by next Tuesday, 2/7. PS3 revisions are due Friday, 2/10 at 11:59 pm. **New policy:** Keep your old solution for PS3. Underneath your old solution, type

  ```
  \paragraph{Revised solution.}
  ```

  and then type your revised solution. This will help make the re-grading process go more smoothly. For future problem sets (such as PS4 revisions), you can just use the command

  ```
  \revisedsolution
  ```

  to indicate the start of your revised solution.

## Exercise 1

Let $A, B, C, D$ be sets. Decide whether each of the following statements is true or false; in each case, either prove the statement or give a counterexample.

(a) $A - (B \cup C) = (A - B) \cup (A - C)$

(b) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

(c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Solution.

(a) The statement $A - (B \cup C) = (A - B) \cup (A - C)$ is false. As a counterexample, consider the sets

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{2\} \\ C &= \{3\}. \end{aligned}$$

Then $A - (B \cup C) = \{1\}$ whereas $(A - B) \cup (A - C) = \{1, 3\} \cup \{1, 2\} = \{1, 2, 3\}$.

(b) The statement $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is also false. The intuition is that an element in $C \times B$ is guaranteed to be in the RHS, but not guaranteed to be in the LHS. So, we construct a counterexample where this is the case.

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then

$$(A \times B) \cup (C \times D) = \{(1,2)\} \cup \{(3,4)\} = \{(1,2), (3,4)\}.$$

On the other hand,

$$(A \cup C) \times (B \cup D) = \{1,3\} \times \{2,4\} = \{(1,2), (1,4), (3,2), (3,4)\}.$$

(c) The statement $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ is true, and we prove this using double inclusion.

First, we'll show that the LHS is contained in the RHS. Let $(x, y) \in (A \times B) \cap (C \times D)$. Since $(x, y) \in A \times B$, we know that $x \in A$ and $y \in B$. Similarly, $(x, y) \in (C \times D)$, so $x \in C$ and $y \in D$. Therefore $x \in A \cap C$ and $y \in B \cap D$, so $(x, y) \in (A \cap C) \times (B \cap D)$.

Second, we'll show that the RHS is contained in the LHS. Let $(u, v) \in (A \cap C) \times (B \cap D)$. Then $u$ is in both $A$ and $C$, and $v$ is in both $B$ and $D$. Therefore $(u, v)$ is in both $A \times B$ and $C \times D$, so indeed $(u, v) \in (A \times B) \cap (C \times D)$.

## Exercise 2

For each of the following relations defined on $\mathbb{Z}$, determine whether it is an equivalence relation. If it is, determine the equivalence classes.

(a) $x \sim y$ if $x \neq y$

(b) $x \sim y$ if $xy > 0$

(c) $x \sim y$ if $x \mid y$ or $y \mid x$

Solution.

(a) The relation "$x \sim y$ if $x \neq y$" is not an equivalence relation because it does not satisfy *reflexivity*.

(b) The relation "$x \sim y$ if $xy > 0$" is *almost* an equivalence relation. However, it fails the *reflexivity* condition for the element $0 \in \mathbb{Z}$: Since $0 \cdot 0 = 0$ is not greater than 0, $0 \not\sim 0$.

(c) The relation $x \sim y$ if $x \mid y$ or $y \mid x$ satisfies the symmetry condition, but fails the *reflexivity* and *transitivity* conditions. Since no integer divides 0 (not even 0), $0 \not\sim 0$, so reflexivity fails. Furthermore, transitivity fails in many cases; for example, $2 \sim 6$ and $6 \sim 3$ (since $2 \mid 6$ and $3 \mid 6$), but $2 \nmid 3$ and $3 \nmid 2$, so $2 \not\sim 3$. So this is also not an equivalence relation.