MAT 108: Problem Set 5

(ADD NAME)

Due 2/14/23 at 11:59 pm on Canvas

Reminders:

- **No Monday 1-2 office hours with Dr. Zhang on 2/13.** TA office hours on 2/13 and Thursday, 2/16 office hours are as usual.
- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS4 by next Tuesday, 2/14. PS4 revisions are due Friday, 2/17 at 11:59 pm. Underneath your old solution, type

  ```
  \revisedsolution
  ```

  and then type your revised solution.

Exercise 1

Consider the set $\mathbb{Z}/7\mathbb{Z}$, equipped with the operations addition ($+$) and multiplication ($\cdot$). We also have division, defined as a function

$$\text{division} : \mathbb{Z}/7\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z} - \{[0]\}) \to \mathbb{Z}/7\mathbb{Z},$$

which you will be able to define by $(x, y) \mapsto xy^{-1}$ after you've completed part (a) below.

Remark. For readability reasons, we will stop using the notation $[x]$ to represent the equivalence class of $x \in \mathbb{Z}$ under the "mod 7" relation. In real life, mathematicians just write "2" for $[2] \in \mathbb{Z}/7\mathbb{Z}$ if the context is clear.

(a) For each of the elements of $\mathbb{Z}/7\mathbb{Z} - \{0\}$, determine its (multiplicative) inverse. (Fill in the table provided.)

(b) Fill in the addition table provided below.

(c) Fill in the multiplication table provided below.

(d) An element $m \in \mathbb{Z}$ is called a *square* if there exists some $n \in \mathbb{Z}$ such that $n \cdot n = m$. We can make the same definition in $\mathbb{Z}/7\mathbb{Z}$. Which elements of $\mathbb{Z}/7\mathbb{Z} - \{0\}$ are *squares*? (Fill in the chart below.)

(0 is indeed a square, but it's extra special, so we consider it separately.)

Remark. Just to be clear, this is an exploration problem. You don't need to write any proofs; just work out the calculations and fill in the tables below!

SOLUTION.

(a)

Element of $\mathbb{Z}/7\mathbb{Z} - \{0\}$	Inverse element
1	
2	
3	
4	
5	
6	

(b)

	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

(c)

	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

(d)

Squares:	
Non-squares:	

Exercise 2

Let $x, y \in \mathbb{R}_{>0}$. Show that if $x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Remark. Remember that you have many properties and axioms to use from Sections 8.1 and 8.2. We didn't explicitly cover them in class if they come from axioms that $\mathbb{Z}$ also satisfied; in that case, the proofs for $\mathbb{Z}$ and for $\mathbb{R}$ are identical.

SOLUTION.

Exercise 3

In class, we talked about upper bounds (Section 8.4). In this problem, you will prove some analogous facts for lower bounds.

(a) Let $B \subseteq \mathbb{R}$ be a nonempty subset. Give a precise definition for the *infimum* $\inf(B)$ of B, i.e. the *greatest lower bound* for B.

(b) Give a definition for $\min(B)$, the *smallest element* of B. (Note that in class we only defined this function for subsets of $\mathbb{Z}$; your definition should be very similar.) Then, prove the following analogue to Proposition 8.49:

Proposition. Let $A \subseteq \mathbb{R}$ be nonempty. If $\inf(A) \in A$, then $\inf(A) = \min(A)$. Conversely, if A has a smallest element, then $\min(A) = \inf(A)$ and $\inf(A) \in A$.

SOLUTION.