MAT 108: Problem Set 6

(ADD NAME)

Due 2/21/23 at 11:59 pm on Canvas

Reminders:

- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS5 by next Tuesday, 2/21. PS4 revisions are due Friday, 2/24 at 11:59 pm. Underneath your old solution, type

  `\revisedsolution`

  and then type your revised solution.

Remark. The word **map** is sometimes used interchangeably with **function**. For example, $e : \mathbb{Z} \to \mathbb{R}$ is a *map* from $\mathbb{Z}$ to $\mathbb{R}$. We can also say that "*e maps* $1_\mathbb{Z}$ to $1_\mathbb{R}$," meaning that the function e sends $1_\mathbb{Z}$ to $1_\mathbb{R}$.

Remark. For Exercises 1 and 2 below, we view $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/6\mathbb{Z}$ as sets with the structure of addition $(+)$. Note that 0 is the additive identity in each case.

Exercise 1

In class, we discussed *injectivity* and *embeddings*.

Definition. Let A and B be sets. A function $f : A \to B$ is *injective* if for every $b \in \operatorname{im}(f)$, there is a unique $a \in A$ such that $f(a) = b$.

In this case, we may say that f maps A *into* B. We may also write $f : A \hookrightarrow B$.

We say that an injective function $f : A \to B$ is an *embedding* if it allows us to identify A with the subset $f(A) \subset B$ in a way that the structures of A and B agree. For example, $e : \mathbb{Z} \hookrightarrow \mathbb{R}$ is an embedding because it preserves addition, multiplication, and ordering.

In other words, an *injection* is a statement about a function f from one set to another. If the sets have extra structure which is preserved by f, then f may be called an *embedding*.

(a) Create addition and multiplication tables for $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/6\mathbb{Z}$.

To typeset these, modify the tables we used in the previous PS.

(b) Define *embeddings* $i : \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ and $j : \mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$.

Make sure you check for yourself that your map is actually an embedding, i.e. respects addition. You do not need to write down a proof that your maps are embeddings, though.

(c) Define an embedding $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ by explicitly stating where each element is sent under the function f.

Again, make sure to check for yourself that your map is actually an embedding. Typeset this as an organized table, by modifying the templates used in the previous PS.

Solution.

(a) Below are the addition and multiplication tables for $\mathbb{Z}/2\mathbb{Z}$:

$(\mathbb{Z}/2\mathbb{Z}, +)$	0	1
0	0	1
1	1	0

$(\mathbb{Z}/2\mathbb{Z}, \cdot)$	0	1
0	0	0
1	0	1

Below are the addition and multiplication tables for $\mathbb{Z}/3\mathbb{Z}$:

$(\mathbb{Z}/3\mathbb{Z}, +)$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$(\mathbb{Z}/3\mathbb{Z}, \cdot)$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Below are the addition and multiplication tables for $\mathbb{Z}/6\mathbb{Z}$:

$(\mathbb{Z}/6\mathbb{Z}, +)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$(\mathbb{Z}/6\mathbb{Z}, \cdot)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(b) Define $i : \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ as

$$0 \mapsto 0$$
$$1 \mapsto 3.$$

Define $j : \mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ as

$$0 \mapsto 0$$
$$1 \mapsto 2$$
$$2 \mapsto 4.$$

(c) Define the function $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ as indicated by the following chart:

$(x, y) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
$f(x, y) \in \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5

Exercise 2

In this exercise, you will learn about *surjectivity* and *projections*.

Definition. Let A and B be sets. A function $g : A \to B$ is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $g(a) = b$.

In this case, we may say that g maps A *onto* B. We may also write $g : A \twoheadrightarrow B$.

(Take a moment to compare the definition of *surjectivity* to the definition of *injectivity*.)

We will talk about the definition of a *projection* later in this course, after we've carefully talked about compositions of maps. But for now, you may use the following alternate definition:

Definition. A surjective map $\pi : A \twoheadrightarrow B$ is a *projection* if there exists and embedding $\iota : B \hookrightarrow A$ such that $\pi \circ \iota = \mathrm{id}_A$.

Here, $\circ$ indicates, as usual, function composition. For example, you'd compute the composite function $\pi \circ \iota$ via the two-step process

$$A \xrightarrow{\iota} B \xrightarrow{\pi} A.$$

The map id_A is the *identity map on* A, i.e. the map $a \mapsto a$ for all $a \in A$.

(a) Define a projection $p : \mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ and prove that it is indeed a projection.

(b) Define a projection $q : \mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$ and prove that it is indeed a projection.

(c) In the previous exercise, you defined an injective function f. Note that it is also surjective (if you did it correctly). Describe the functions $p \circ f$ and $q \circ f$.

However you decide to describe these functions, whether explicitly or descriptively, the reader should be able to compute the image of every element in the domain after having read your description.

SOLUTION.

(a) We may define $p : \mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ as shown in the following chart:

$k \in \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
$p(k) \in \mathbb{Z}/2\mathbb{Z}$	0	1	0	1	0	1

This is indeed a projection, since $p \circ i = \mathrm{id}$:

$$p \circ i(0) = p(0) = 0$$
$$p \circ i(1) = p(3) = 1.$$

(b) Similarly, we define $q : \mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$ according to the following chart:

$k \in \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
$q(k) \in \mathbb{Z}/3\mathbb{Z}$	0	0	1	1	2	2

This is a projection because $q \circ j = \text{id}$ as well:

$$q \circ j(0) = q(0) = 0$$
$$q \circ j(1) = q(2) = 1$$
$$q \circ j(2) = q(4) = 2.$$

(c) We can compute the function $p \circ f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ explicitly:

$(x, y) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
$f(x, y) \in \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
$p \circ f(x, y) \in \mathbb{Z}/2\mathbb{Z}$	0	1	0	1	0	1

We can compute the function $q \circ f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ explicitly in the same way:

$(x, y) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
$f(x, y) \in \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
$q \circ f(x, y) \in \mathbb{Z}/3\mathbb{Z}$	0	0	1	1	2	2

Remark. Intriguingly, these composite maps are *not* the projections we might expect, i.e. described as a map $(x, y) \mapsto x$ or $(x, y) \mapsto y$. This is related to the difference between *embeddings* and *quotients*.

Exercise 3

The *Pigeonhole Principle* is a powerful tool in mathematics. And yes, this is a technical term.

Lemma (Pigeonhole Principle)**.** (Let $n, m \in \mathbb{N}$.) If you have n pigeons and m cubby holes, and $n > m$, then no matter how you put the pigeons in the holes, there will be a hole with more than one pigeon.

Use the Pigeonhole Principle to show that there does not exist an injective map

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \hookrightarrow \mathbb{Z}/15\mathbb{Z}.$$

Solution.

Pick any function $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \hookrightarrow \mathbb{Z}/15\mathbb{Z}$.

We first count the number of elements $(x, y, z) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. There are 2 choices for x, 3 for y, and 5 for z, so there must be $2 \cdot 3 \cdot 5 = 30$ elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. These are mapped by f to the 15 elements of $\mathbb{Z}/15\mathbb{Z}$.

By the Pigeonhole Principle, there will be some element $k \in \mathbb{Z}/5\mathbb{Z}$ such that more than one element of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ maps to k under f. Therefore f is not injective.

Exercise 4

We are all used to the *Euclidean norm* on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, denoted $\|\cdot\|$. This is the measure of distance where the distance between two points $(x_0, y_0), (x_1, y_1)$ is given by

$$\|(x_1, y_1) - (x_0, y_0)\| = \sqrt{(x_1 - x_0)^2 - (y_1 - y_0)^2}.$$

We can define a different metric on $\mathbb{R}^2$, called the *Manhattan norm*, denoted $\|\cdot\|_1$. The *Manhattan distance* between $(x_0, y_0), (x_1, y_1)$ is given by

$$\|(x_1, y_1) - (x_0, y_0)\|_1 = |x_1 - x_0| + |y_1 - y_0|.$$

Remark. The subscript "1" has to do with the fact that $\|\cdot\|_1$ is called the L_1-norm and comes from a whole family of norms, one for each natural number, plus infinity. The Euclidean norm is the L_2-norm.

(a) Prove that "Manhattan distance" is actually a distance function.

(b) Prove that the Manhattan distance between any two points is always at least the Euclidean distance between them, i.e.

$$\|(x_1, y_1) - (x_0, y_0)\|_1 \geq \|(x_1, y_1) - (x_0, y_0)\|.$$

for any two points $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$.

SOLUTION.

(a) To check that $\|\cdot\|_1$ is a distance measure, we need to check the three criteria for distance functions:

(i) *The distance between two points $p = (x_0, y_0)$ and $q = (x_1, y_1)$ is nonnegative, and is equal to 0 if and only if $p = q$.*
First of all, since the absolute value function is takes nonnegative values only, the Manhattan distance between p and q is the sum of two nonnegative numbers, and is therefore also nonnegative.
If $p = q$, then $x_0 = x_1$ and $y_0 = y_1$, so the distance from p to q is $\|q - p\|_1 = 0 + 0 = 0$.
If $p \neq q$, then $x_0 \neq x_1$ or $y_0 \neq y_1$, so at least one of $|x_1 - x_0|$ or $|y_1 - y_0|$ is > 0. Therefore their sum is also > 0.

(ii) *The distance from p to q is the same as the distance from q to p.*
Since $|x_1 - x_0| = |-(x_0 - x_1)| = |x_0 - x_1|$ and similarly $|y_1 - y_0| = |y_0 - y_1|$,

$$\|q - p\|_1 = |x_1 - x_0| + |y_1 - y_0| = |x_0 - x_1| + |y_0 - y_1| = \|p - q\|_1,$$

so Manhattan distance is symmetric.

(iii) *The distance between points p and $r = (x_2, y_2)$ is at most the sum of the distance from p to q and the distance from q to r.*
The sum of the distances from p to q and from q to r is

$$\begin{aligned}\|q - p\|_1 + \|r - q\|_1 &= \|(x_1, y_1) - (x_0, y_0)\|_1 + \|(x_2, y_2) - (x_1, y_1)\|_1 \\ &= (|x_1 - x_0| + |y_1 - y_0|) + (|x_2 - x_1| + |y_2 - y_1|).\end{aligned}$$

After rearranging the summands above, we have

$$= (|x_1 - x_0| + |x_2 - x_1|) + (|y_1 - y_0| + |y_2 - y_1|)\,,$$

which by the Triangle Inequality for $|\cdot|$ is

$$\geq |x_2 - x_0| + |y_2 - y_0| = \|(x_2, y_2) - (x_0, y_0)\|_1 = \|r - p\|_1.$$

Therefore $\|q - p\|_1 + \|r - q\|_1 \geq \|r - p\|_1$.

(b) Given two points (x_0, y_0) and (x_1, y_1), consider a third point (x_0, y_1). The three points form a right triangle, with legs of length $|x_1 - x_0|$ and $|y_1 - y_0|$. (The triangle would be *degenerate*, i.e. completely flat, if $x_0 = x_1$ or $y_0 = y_1$.)

By the Triangle Inequality for Euclidean distance, the sum of the lengths of the legs of the right triangle is greater than or equal to the length of the hypotenuse. Therefore

$$\|(x_1, y_1) - (x_0, y_0)\|_1 = |x_1 - x_0| + |y_1 - y_0| \geq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \|(x_1, y_1) - (x_0, y_0)\|.$$