MAT 108: Problem Set 7

(ADD NAME)

Due 2/28/23 at 11:59 pm on Canvas

Reminders:

- Exam 2 is Wednesday, March 1, in class. It will cover all the material we covered in February, including all the material on PS 4–7.
  - To study for this exam, I recommend solving problems from the book, and also making sure you are able to solve previous PS exercises.
  - Once again, style will be very important. If you lost style points on Exam 1, I urge you to look at the comments on your graded Exam 1 and ask me or Hans if you aren't sure why you lost style points.
  - Discussion on Tuesday, 2/28 will a review session where you'll have the opportunity to practice solving problems similar in flavor to those on the exam.
- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS6 by next Tuesday, 2/28. PS5 revisions are due Friday, 3/3 at 11:59 pm. Underneath your old solution, type

  `\revisedsolution`

  and then type your revised solution.

Exercise 1

Prove that limits of sequences of real numbers are unique.

Hint: In other words, prove that if (x_k) converges to L_1 and to L_2, then $L_1 = L_2$. (This is a uniqueness statement.)

Solution.

Let (x_k) be a sequence of real numbers that converges to both L_1 and L_2. To show that $L_1 = L_2$, it suffices to show that given any $\varepsilon > 0$, $|L_1 - L_2| < \varepsilon$ (Proposition 10.11, discussed in class).

Let $\varepsilon > 0$. Since (x_k) converges to L_1, there is some N_1 such that for all $n \geq N_1$,

$$|x_n - L_1| < \frac{\varepsilon}{2}.$$

Since (x_k) also converges to L_2, there is some N_2 such that for all $n \geq N_2$,

$$|x_n - L_2| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$|L_1 - L_2| = |L_1 - x_n + x_n - L_2| \leq |L_1 - x_n| + |x_n - L_2| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the first inequality is the triangle inequality for absolute value.

Exercise 2

The **Monotone Convergence Theorem** is a powerful tool in analysis. It states that

Every monotonic bounded sequence converges.

In class, we proved that every *increasing* bounded sequence converges (Theorem 10.19). Prove the analogous statement to Theorem 10.19 for *decreasing* bounded sequences.

(Write down the precise statement you are proving before you prove it. The Proposition environment has been included in the solution area below.)

Solution.

Proposition. Every decreasing bounded sequence converges.

Proof. Let $(x_k)_{k=1}^{\infty}$ be a decreasing and bounded sequence. By the Completeness Axiom, since the set $\{-x_k : k \in \mathbb{N}\}$ is bounded, it has a supremum s. Then $-s$ is the infimum of the set $X = \{x_k : k \in \mathbb{N}\}$.

We claim that $-s = \lim_{k\to\infty} x_k$. To prove this, let $\varepsilon > 0$. Then $-s + \varepsilon$ is *not* an upper bound for X, since $-s + \varepsilon > -s$ and $-s$ is the greatest lower bound. Therefore there exists $N \in \mathbb{N}$ such that $x_N < -s + \varepsilon$. But (x_k) is decreasing, so for all $n > N$,

$$-s \leq x_n \leq x_N \leq -s + \varepsilon,$$

i.e. $|x_n - (-s)| < \varepsilon$. In other words, $-s = \lim_{k\to\infty} x_k$. □

Exercise 3

We can restate the Monotone Convergence Theorem as follows:

If a sequence is monotone and bounded, then it converges.

In this exercise, you will see that monotone, bounded sequences are "special" within the set of convergence sequences.

(a) Prove the following partial converse to the Monotone Convergence Theorem:

Proposition. If a sequence converges, then it is bounded.

(b) Notice that the partial converse does not conclude that the converging sequence must also be monotone. Give an example of a sequence that converges but is not monotone. *Make sure you prove that your sequence indeed converges and is indeed not monotone!*

Solution.

(a) Let (x_k) be a convergent sequence, and let $L = \lim_{k\to\infty} x_k$. Then (for $\varepsilon = 1$,) there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < 1$.

Define two sets

$$A = \{x_k \ : \ k < N\} \qquad \text{and} \qquad B = \{x_k \ : \ k \geq N\}$$

so that the set of all terms in (x_k) is equal to $A \cup B$. The set A is finite, so it is bounded below by $\min A$ and above by $\max A$. By our choice of N, for all $b \in B$, $L - 1 \leq b \leq L + 1$.

Thus for all $x_k \in A \cup B$,

$$\min\{\min A, L-1\} \leq x_k \leq \max\{\max A, L+1\},$$

so the sequence (x_k) is bounded.

(b) Consider the sequence $(x_k)_{k=1}^{\infty}$ where

$$x_k = \frac{(-1)^k}{k}.$$

This sequence begins with the terms $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots$. The terms alternate between positive and negative real numbers, and therefore alternates between increasing and decreasing. That is, for $k \geq 2$, if k is even, then $x_k > 0 > x_{k-1}$; if k is odd, then $x_k < 0 < x_{k-1}$.

Nevertheless, we show the sequence converges to 0. Let $\varepsilon > 0$. Since $\mathbb{N}$ is unbounded, there exists some N such that $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$|x_n - 0| = \left|\pm\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon,$$

so $\lim_{k\to\infty} x_k = 0$.

Exercise 4

Definition. An integer n is a *perfect square* if $n = m^2$ for some $m \in \mathbb{Z}$.

Prove that if $r \in \mathbb{N}$ is not a perfect square, then $\sqrt{r}$ is irrational. *Hint: Emulate the proof of Proposition 11.10, which states that $\sqrt{2}$ is irrational.*

Solution.

Fix $r \in \mathbb{N}$ and suppose that r is not a perfect square. Suppose, by way of contradiction, that $\sqrt{r} \in \mathbb{Q}$, so that we may write

$$\sqrt{r} = \frac{m}{n}$$

with $m, n \in \mathbb{N}$, in reduced terms (where m and n have no common factors). Then

$$r = \frac{m^2}{n^2} = \frac{m}{n} \cdot \frac{m}{n}$$

so we may write

$$\frac{m}{n} = \frac{rn}{m}$$

by multiplying both sides by $\frac{n}{m} \neq 0$.

Since $\frac{m}{n}$ is in lowest terms, Proposition 11.5 tells us that $n \mid m$, i.e. $\frac{m}{n} = \sqrt{r} \in \mathbb{Z}$. But $\sqrt{r}^2 = r$ is not a perfect square, so we have a contradiction.