MAT 108: Problem Set 8

Solutions

Due 3/7/23 at 11:59 pm on Canvas

Reminders:

- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS7 by next Tuesday, 3/7. PS7 revisions are due Friday, 3/10 at 11:59 pm. Underneath your old solution, type

  `\revisedsolution`

  and then type your revised solution.

Exercise 1

Determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}.$$

Be careful. You may use the formula for the sum of a geometric series, but be careful with indices, and make sure you know what the "first term" is.

SOLUTION.

Recall that the sum of the geometric series $\sum_{n=0}^{\infty} ar^n$ is $\frac{a}{1-r}$. We my rewrite the given series as

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2^n}{3^n}\right) = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n,$$

so the sum of the series is

$$\frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1.$$

Exercise 2

Prove Proposition 12.4:

Proposition. Let $(d_k)_{k=1}^{\infty}$ be a sequence of digits. Then $\sum_{j=1}^{\infty} d_j \cdot 10^{-j}$ converges.

Remark. The digits in this exercise are base 10 digits, i.e. for all j, $d_j \in \{0, 1, \ldots, 9\}$.

Solution.

For all j, $0 \leq d_j \leq 10$. Therefore we may use the Comparison Test and compare our series with the geometric series

$$\sum_{j=1}^{\infty} 10 \cdot 10^{-j}.$$

The common ratio r for this series is $r = \frac{1}{10}$, so $|r| < 1$. Therefore this geometric series converges. Since $d_j 10^{-j} \leq 10 \cdot 10^{-j}$ for all j, the given series $\sum_{j=1}^{\infty} d_j \cdot 10^{-j}$ converges as well.

Exercise 3

Write the following decimal expansions as fractions.

(a) $5.\overline{6} = 5.66666...$

(b) $0.346\overline{127} = 0.346127127127...$

Solution.

(a) We first write the decimal expansion as a geometric series:

$$5.\overline{6} = 5 + \sum_{j=1}^{\infty} 6 \cdot 10^{-j}.$$

The series

$$\sum_{j=1}^{\infty} 6 \cdot 10^{-j} = \sum_{j=1}^{\infty} 6 \cdot \left(\frac{1}{10}\right)^{j} = \sum_{j=1}^{\infty} \frac{6}{10} \cdot \left(\frac{1}{10}\right)^{j-1} = \sum_{j=0}^{\infty} \frac{6}{10} \cdot \left(\frac{1}{10}\right)^{j}$$

is geometric, with $a = \frac{6}{10} = \frac{3}{5}$ and $r = \frac{1}{10}$. Therefore its sum is $\frac{a}{1-r} = \frac{3}{5} \cdot \frac{10}{9} = \frac{2}{3}$. So

$$5.\overline{6} = 5 + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}.$$

(b) Once again, we first handle the repeating part of the decimal expansion:

$$0.000\overline{127} = 10^{-3} \cdot 0.\overline{127} = 10^{-3} \cdot \sum_{j=1}^{\infty} \frac{127}{1000^j} = 10^{-6} \cdot \sum_{j=0} 127 \cdot \frac{1}{1000^j}.$$

Here, the leading term is $a = 127 \cdot 10^{-6}$ and the common ratio is $r = \frac{1}{1000}$, so the sum of the series $\frac{a}{1-r}$ is

$$= a \cdot (1-r)^{-1} = (127 \cdot 10^{-6}) \cdot \left(1 - \frac{1}{1000}\right)^{-1} = (127 \cdot 10^{-6}) \cdot \frac{1000}{999} = \frac{127}{999} \cdot 10^{-3} = \frac{127}{999000}.$$

Then

$$0.346\overline{127} = \frac{346}{1000} + \frac{127}{999000} = \frac{346 \cdot 999 + 127}{999000} = \frac{345781}{999000}.$$

Exercise 4

Express the following fractions as decimals.

(a) $\frac{71}{13}$

(b) $\frac{34}{31}$

SOLUTION.

(a) Using the division algorithm, we find that

$$\begin{aligned}
71 &= 5 \cdot 13 + 6 \\
60 &= 4 \cdot 13 + 8 \\
80 &= 6 \cdot 13 + 2 \\
20 &= 1 \cdot 13 + 7 \\
70 &= 5 \cdot 13 + 5 \\
50 &= 3 \cdot 13 + 11 \\
110 &= 8 \cdot 13 + 6.
\end{aligned}$$

At this point, we know that our algorithm will repeat. Therefore

$$\frac{71}{13} = 5.\overline{461538}.$$

(b) Again using the division algorithm, we have

$$\begin{aligned}
34 &= 1 \cdot 31 + 3 \\
30 &= 0 \cdot 31 + 30 \\
300 &= 9 \cdot 31 + 21 \\
210 &= 6 \cdot 31 + 24 \\
240 &= 7 \cdot 31 + 23 \\
230 &= 7 \cdot 31 + 13 \\
130 &= 4 \cdot 31 + 6 \\
60 &= 1 \cdot 31 + 29 \\
290 &= 9 \cdot 31 + 11 \\
110 &= 3 \cdot 31 + 17 \\
170 &= 5 \cdot 31 + 15 \\
150 &= 4 \cdot 31 + 26 \\
260 &= 8 \cdot 31 + 12 \\
120 &= 3 \cdot 31 + 27 \\
270 &= 8 \cdot 31 + 22 \\
220 &= 7 \cdot 31 + 3,
\end{aligned}$$

at which point we know the algorithm will repeat. Thus

$$\frac{34}{31} = 1.\overline{096774193548387}.$$