MAT 108: Problem Set 9

Solutions

Due 3/14/23 at 11:59 pm on Canvas

Reminders:

- Your homework submission must be typed up in full sentences, with proper mathematical formatting. Handwritten homework submissions will receive a score of 0. Solutions containing incomplete sentences or poor formatting will lose points.
- You will receive feedback on PS8 by next Tuesday, 3/14. PS8 revisions are due Friday, 3/17 at 11:59 pm. Underneath your old solution, type

  `\revisedsolution`

  and then type your revised solution.
- All assignments for this course, except the final exam, must be submitted by Friday, 3/17 at 11:59 pm; no extensions will be possible beyond this time.

Grading for this problem set This problem set will be graded for *completion*: an honest attempt given to solve the problem will be given full marks. The solutions will be posted two days after the set is due so that you can verify your own answers.

Exercise 1

Describe an algorithm for "counting" the countable set $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. In other words, describe how one could construct a bijective function $\mathbb{N} \to \mathbb{N}^3$.

Your description doesn't need to be 100% rigorous; this would take a long time to write down. However, your description needs to be clear enough so that a hypothetical classmate who hasn't thought about this problem would be able to understand how to count $\mathbb{N}^3$, and understand why your counting method would reach any given element in $\mathbb{N}^3$ in finite time.

Hint. Think about how we "counted" $\mathbb{N} \times \mathbb{N}$ or $\mathbb{Z} \times \mathbb{Z}$ in class.

SOLUTION.

Here's one way to count $\mathbb{N}^3$. For each $k \in \mathbb{N}$ where $k \geq 3$, let $S_k \subset \mathbb{N}^3$ denote the subset of triples (a, b, c) where $a + b + c = k$. For example,

$$S_3 = \{(1, 1, 1)\}$$
$$S_4 = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$$
$$S_5 = \{(3, 1, 1), (1, 1, 3), (1, 3, 1), (2, 2, 1), (2, 1, 2), (1, 2, 2)\}$$

For any fixed k, the size of the set S_k is finite, and so we can count it in finitely many steps. To see that S_k is finite, notice that there are at most k choices for values in each of the three components (and furthermore, many triples we get this way aren't even in S_k), so card $S_k \leq k^3$.

Since $\mathbb{N}^3 = \bigcup_{k=3}^{\infty}$, we can count $\mathbb{N}^3$ by first counting everything in S_3, then everything in S_4, then S_5, then S_6, and so on.

Remark. This solution comes from thinking of $\mathbb{N}^3$ as all the *lattice points* in a 3D space whose coordinates are positive integers. One might call this region of xyz-space the *first octant*. Each batch we count is a slice of the octant.

Exercise 2

Prove that the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ has the same cardinality as $\mathbb{R}$.

Hint. You need to find a bijection between the two sets. Do you know of a function from calculus class that gives a bijection $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$?

Remark. Once we know that one open interval has the same cardinality as $\mathbb{R}$, by scaling and translating using a linear function, we can show that any open interval $(a, b) \subset \mathbb{R}$ (where $a < b$) has the same cardinality as $\mathbb{R}$.

SOLUTION.

Consider tangent function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. It is injective, since if $\tan\theta_1 = \tan\theta_2$, then $|\theta_1 - \theta_2|$ is a multiple of π; since the domain is $(-\frac{\pi}{2}, \frac{\pi}{2})$, that multiple must be 1, and $\theta_1 = \theta_2$. This function is also surjective, since for any real number $r \in \mathbb{R}$, if we let $\theta = \arctan(r)$, then $\tan(\theta) = r$. Therefore tan is bijective, and so card $(-\frac{\pi}{2}, \frac{\pi}{2}) =$ card $\mathbb{R}$. □

Exercise 3

Prove that for each $n \in \mathbb{N}$, card $\mathcal{P}([n]) =$ card $[2^n]$.

SOLUTION.

Here are two solutions to this exercise.

Solution 1 We first define a function $f_n : \mathcal{P}([n]) \to \mathbb{Z}_2^n$, using the following procedure. Given $S \subset [n]$, set the i-th bit of $h_n(S)$ to be

$$\begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S. \end{cases}$$

The function f_n clearly has an inverse: given a sequence of bits $w \in \mathbb{Z}_2^n$, construct a subset S of $[n]$ by throwing in i if and only if the i-th bit of w is a 1. Since f_n has an inverse, it is bijective.

Now every $w \in \mathbb{Z}_2^n$ is a binary string that represents some number in $\{0, 1, \ldots, 2^n - 1\}$. Define a function $g_n : \mathbb{Z}_2^n \to [2^n]$ by sending w to the integer it represents (as a binary string), plus one. Now g_n is also a bijection, since we can define an inverse by sending $k \in 2^n$ to the binary representation of $k - 1$.

Then $g_n \circ f_n$ is a bijection from $\mathcal{P}([n]) \to [2^n]$. □

Solution 2 We will prove the statement by induction on n. Let $Q(n)$ be the statement

$$\text{card } \mathcal{P}([n]) = \text{card } [2^n], \text{ i.e. there exists a bijective function } f_n : \mathcal{P}([n]) \to [2^n].$$

In the base case, $n = 1$, and $\mathcal{P}([1]) = \{\emptyset, \{1\}\}$. Then

$$\begin{aligned} f_1 : \mathcal{P}([1]) &\to [2] \\ \emptyset &\mapsto 1 \\ \{1\} &\mapsto 2 \end{aligned}$$

is a bijection.

Now assume that for a given n, there is a bijective function $f_n : \mathcal{P}([n]) \to [2^n]$. We construct a bijective function $f_{n+1} : \mathcal{P}([n+1]) \to [2^{n+1}]$ as follows.

Observe that if a subset $S \subset [n+1]$ does *not* contain $n+1$, then it is also a subset of $[n]$, and so $f_n(S)$ is defined.

For a subset $S \subset [n+1]$, let

$$f_{n+1}(S) = \begin{cases} f_n(S) & \text{if } n+1 \notin S \\ f_n(S - \{n+1\}) & \text{if } n+1 \in S \end{cases}$$

In other words, the subsets of $[n+1]$ that do not contain $n+1$ are mapped to

$$[2^n] = \{1, 2, \ldots, 2^n - 1, 2^n\},$$

and then the subsets that do contain $n+1$ are mapped to

$$[2^{n+1}] - [2^n] = \{2^n + 1, 2^n + 2, \ldots, 2^{n+1} - 1, 2^{n+1}\}.$$

We now show that f_{n+1} is bijective.

We first show that f_{n+1} is injective. Suppose $S, S' \subset [n+1]$ satisfy $f_{n+1}(S) = f_{n+1}(S')$. Then either (a) $f_{n+1}(S) \in [2^n]$ or (b) $f_{n+1}(S) \in [2^{n+1}] - [2^n]$. If we are in case (a), then S and S' both do not contain $n+1$; then, since f_n is injective, we have $S = S'$. If we are in case (b), then S and S' both contain $n+1$; then $S - \{n+1\}$ and $S' - \{n+1\}$ must again be the same set T because f_n is injective; then $S = T \cup \{n+1\} = S'$ as well.

We now show that f_{n+1} is surjective. As stated previously, every subset $T \subset [n]$ is also a subset of $[n+1]$. There are $2^n = \text{card } \mathcal{P}([n])$ of these. The set $\{T \cup \{n+1\} : T \in \mathcal{P}([n])\}$ also has 2^n elements, as they are clearly in bijection with $\mathcal{P}([n])$. We therefore have a total of $2^n + 2^n = 2^{n+1}$ elements that are injectively mapped by f_{n+1} to the set $[2^{n+1}]$, which also has 2^{n+1} elements. Therefore f_{n+1} must be surjective; otherwise, the Pigeonhole Principle would show that the function is not injective.

In summary, we have constructed a bijective function $f_{n+1} : \mathcal{P}([n+1]) \to [2^{n+1}]$, so

$$\text{card } \mathcal{P}([n+1]) = \text{card } [2^{n+1}]$$

indeed. □