MAT 108: Final Exam Study Guide

March 14, 2023

1 Important Information

- Location: Walker 1320
- Date: Thursday, March 23rd, 2023
- Time: 6:00 pm – 8:00 pm

2 Topics covered this quarter

- Axiomatic approach
  - axioms
  - if-then statements
  - uniqueness and existence
- Induction
  - natural numbers
  - recursion
- Symbolic logic
  - for all, there exists, negation
  - implications
  - converse, inverse, contrapositive
  - proof by contradiction
- Set theory
  - sets, functions between sets
  - unions, intersections, complements
  - De Morgan's laws
  - Cartesian products
- Functions
  - injections, surjections

  - bijections
  - domain, codomain, image, preimage

- Equivalence relations
  - criteria for equivalence relations
  - $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$, $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$

- Numbers, our main examples
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R} - \mathbb{Q}$
  - Well-ordering principle
  - functions, injections, surjections, bijections
  - upper/lower bounds
  - Completeness Axiom

- Norms and distance functions
  - absolute value and distance between real numbers
  - other norms and how they define distance functions
  - criteria for distance functions / norms

- Limits
  - sequences
  - convergence, divergence
  - $\varepsilon - N$ definition of a limit
  - Monotone Convergence Theorem
  - square roots

- Infinite series
  - sequence of terms, sequence of partial sums
  - sum of a series
  - convergence, divergence, Comparison Test
  - decimal and binary expansions
  - examples of convergence: geometric series
  - example of divergence: harmonic series

- Cardinality
  - definition of $\operatorname{card}(A)$
  - cardinality for finite sets
  - countable, countably infinite, continuum
  - comparing cardinalities $(\leq, <)$
  - powersets
  - Cantor's diagonalization argument

3 Mock Exam

Solutions to this Mock Exam will be posted by Monday of finals week.

1. Assume the following two *axioms*:

   A1 The area of a planar rectangle with sides $a, b \in \mathbb{R}$ is the product $a \cdot b$.

   A2 The area of two planar figures which intersect at most along edges is the sum of areas of each of the planar figures.

   Use Axioms 1 and 2 to deduce that the area of the triangle with height $h \in \mathbb{R}$ and base $b \in \mathbb{R}$ equals $(b \cdot h)/2$.

2. Prove that for $n \geq 8$,
$$3n^2 + 3n + 1 < 2^n.$$

3. Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

4. Consider the following recursively defined sequence:
$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + 1 \qquad \forall\, n \in \mathbb{N}.$$
   (a) Show that (x_n) is increasing and bounded above.

   (b) Prove that (x_n) converges and find its limit.

5. Consider the set of black and white colorings of the numbers in the interval $[0, 1] \subset \mathbb{R}$:
$$C := \{c : [0, 1] \to \{\text{black}, \text{white}\}\}.$$
   Show that $\operatorname{card} C > \operatorname{card} \mathbb{R}$. *Note the strict inequality.*

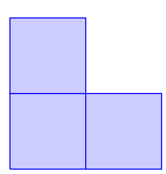
4 More Practice Problems

See your book for more practice problems. I will not be providing solutions for these. Select problems in this list will be covered in Discussion.

1. Let $k \in \mathbb{N}$. Consider a $2^k \times 2^k$ square board divided into equal square tiles of size 1×1, like a chess board. (Thus the $2^k \times 2^k$ board is covered by $(2^k)^2$ tiles.)

   Remove one tile from the $2^k \times 2^k$ board.

   Prove *by induction* that the remaining part of the board can be covered with *triomino* pieces, i.e. pieces of three unit tiles with an L-shape:



2. Prove that for all $n \in \mathbb{N}$, $7 \mid (2^{n+2} + 3^{2n+1})$.

3. Prove that for all $n \in \mathbb{N}$, $5 \mid (11^n - 6)$.

4. Consider the subset $C_0 = [0, 1] \subseteq \mathbb{R}$. Recursively define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3} \right)$$

for $n \geq 1$, where for an interval $A = [a, b]$,

- $A/3$ is the interval $[a/3, b/3]$ and
- $A + 2/3$ is the interval $[a + 2/3, b + 2/3]$.

(a) Describe and draw the sets C_1, C_2, and C_3 as a union of explicit intervals.

(b) Show that the intersection $\bigcap_{n=1}^{\infty} C_n$ is non-empty.

5. Consider the set of real numbers

$$A = \left\{ 3 - \frac{1}{n} \ : \ n \in \mathbb{N} \right\}.$$

Find $\inf A$ and $\sup A$.